Composite and inverse functions

P2.8 · Pure mathematics · revision notes

Functions, their domain and range, composite functions such as $fg(x)$ (which means $f(g(x))$) and inverse functions and their graphs. Questions usually have several parts: find the range, find or solve a composite, find $f^{-1}(x)$ and state its domain. They are often worth 7 to 10 marks in total.

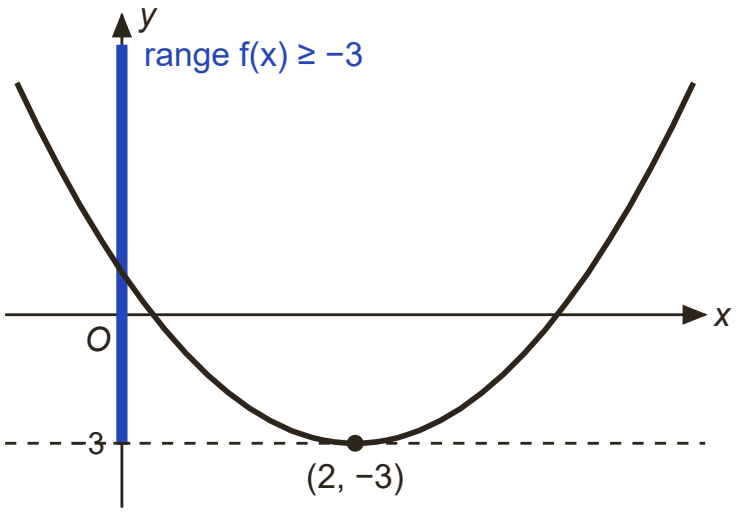
Key facts

- $fg(x) = f(g(x))$: do g first
- $ff^{-1}(x) = f^{-1}f(x) = x$
- Range of f^{-1} = domain of f
- An inverse exists only if f is one-to-one
- $f^2(x) = ff(x)$
- Domain of f^{-1} = range of f
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$
- $x \in \mathbb{R}$ means x can be any real number

Notes

Functions, domain and range

- A function maps each input in its domain to exactly one output. The set of outputs is the range.
- One-to-one: each output comes from exactly one input. Many-to-one: some outputs come from more than one input, e.g. $f(x) = x^2$, $x \in \mathbb{R}$
- Find the range from a sketch: $f(x) = (x - 2)^2 - 3$, $x \in \mathbb{R}$, has range $f(x) \geq -3$



Read the range off the y-axis: the curve takes every value from −3 upwards.

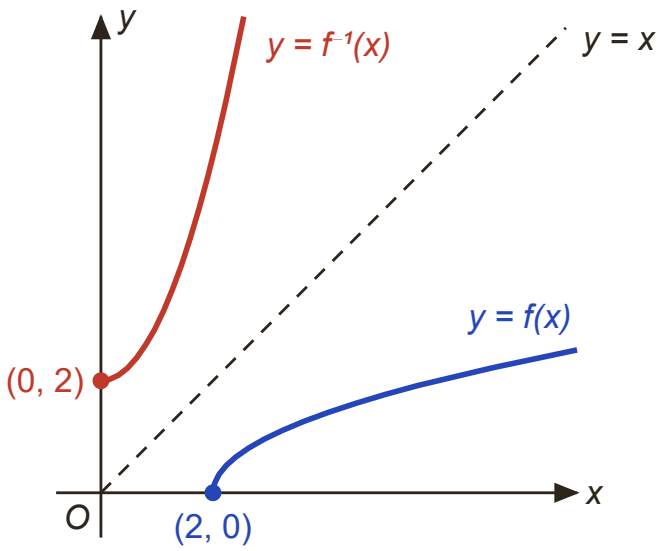
- Write the range in terms of $f(x)$ (or y), not x. Exclude from a domain any values that would mean dividing by zero or square rooting a negative.

Composite functions

- $fg(x)$ means $f(g(x))$: apply g first, then f. The function nearest x acts first.
- Example: $f(x) = 2x + 1$ and $g(x) = x^2$ give $fg(x) = 2x^2 + 1$ but $gf(x) = (2x + 1)^2$. Usually $fg \neq gf$
- $f^2(x)$ means $ff(x)$, not $[f(x)]^2$
- fg can only be formed if the range of g lies within the domain of f
- The range of fg comes from applying f to the range of g, not to every real number: e.g. $g(x) = x^2$, $f(x) = 2x + 1$ gives $fg(x) \geq 1$ Grade A or above only

Inverse functions

- f^{-1} undoes f: if $f(a) = b$ then $f^{-1}(b) = a$, and $ff^{-1}(x) = f^{-1}f(x) = x$
- Only one-to-one functions have an inverse.
- Method: let $y = f(x)$, rearrange to make x the subject, then replace y by x
- Example: $f(x) = \sqrt{x - 2}$, $x \geq 2$, has range $f(x) \geq 0$. From $y = \sqrt{x - 2}$, $x = y^2 + 2$, so $f^{-1}(x) = x^2 + 2$, $x \geq 0$
- The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f
- The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$



f(x) = √(x − 2) and f⁻¹(x) = x² + 2, x ≥ 0: each is the mirror image of the other in y = x.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the range of a function

2 marks | Grade C

1. Complete the square or sketch the graph over the given domain.
2. Find the least (or greatest) value.
3. Write the range using $f(x)$

Example. The function f is defined by $f(x) = x^2 - 6x + 11, x \in \mathbb{R}$. Find the range of f

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Model answer.

$f(x) = (x-3)^2 + 2$ (M1)

$f(x) \geq 2$ (A1)

Evaluate and solve composite functions

5 marks | Grade B

1. For a number: work from the inside out.
2. For an expression: substitute the inner function into the outer one.
3. Set the composite equal to the value and solve, checking the answer is in the domain.

Example. The functions f and g are defined by

$f(x) = 3x - 2, x \in \mathbb{R}$

$g(x) = \dfrac{4}{x+1}, x > -1$

(a) Find $fg(3)$.

(b) Solve $gf(x) = 2$

Model answer.

(a) $g(3) = 1$ (M1)

$fg(3) = f(1) = 1$ (A1)

(b) $gf(x) = \dfrac{4}{3x - 2 + 1}$

$= \dfrac{4}{3x-1}$ (M1)

$\dfrac{4}{3x-1} = 2$, so $3x - 1 = 2$ (M1)

$x = 1$ (A1)

Find an inverse function and its domain

3 marks **Grade B**

1. Write $y = f(x)$ and multiply out any fraction.
2. Collect the x terms on one side and factorise to make x the subject.
3. Write $f^{-1}(x)$, and give its domain as the range of f

Example. The function h is defined by $h(x) = \dfrac{2x+3}{x-1}$, $x > 1$.

(a) Find $h^{-1}(x)$.

(b) State the domain of h^{-1}

Model answer.

(a) $y(x-1) = 2x+3$, so $xy - 2x = y + 3$ and $x(y-2) = y+3$ (M1)

$h^{-1}(x) = \dfrac{x+3}{x-2}$ (A1)

(b) $h(x) = 2 + \dfrac{5}{x-1} > 2$ for $x > 1$, so the domain of h^{-1} is $x > 2$ (B1)

Explain why a function has no inverse

2 marks **Grade B**

1. Show the function is many-to-one, e.g. two inputs with the same output.
2. To make it one-to-one, restrict the domain to one side of the turning point.

Example. The function g is defined by $g(x) = (x+1)^2 - 4$, $x \in \mathbb{R}$.

(a) Explain why g does not have an inverse.

(b) The domain of g is changed to $x \geq k$ so that g has an inverse. State the least possible value of k

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Model answer.

(a) g is many-to-one, e.g. $g(-3) = g(1) = 0$, so it has no inverse. (B1)

(b) $k = -1$ (the x-coordinate of the turning point) (B1)

Don't lose marks

- Doing composite functions in the wrong order: fg means g first.
- Writing the range in terms of x, e.g. '$x \geq 2$' instead of '$f(x) \geq 2$'.
- Confusing $f^{-1}(x)$ with $\dfrac{1}{f(x)}$
- Giving the domain of f as the domain of f^{-1} instead of the range of f
- Using $>$ when a minimum value is actually reached, so the range needs $\geq$

More tips

Memory tricks

- Read $fg(x)$ from right to left: the function nearest x acts first.
- Check an inverse with a number: $f^{-1}(f(a))$ should give back a
- Sketch first for range questions: turning points and asymptotes give the limits.
- Reflection in $y = x$ swaps coordinates: (a, b) on f becomes (b, a) on f^{-1}

Exam technique

- Use the correct notation: $f(x) \geq 2$ for a range, $x \in \mathbb{R}$ for all real numbers, $f^{-1}(x) = \ldots$ for an inverse.
- In multi-part function questions, a range found in an early part is often the domain you need later.
- When sketching f and f^{-1}, draw the line $y = x$ and label where each graph meets the axes.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Evaluate a composite function at a number**
$fg(3)$ means find $g(3)$ first, then put the answer into f

Grade D — **Find a composite function algebraically**
Replace x in f with the whole expression for $g(x)$: $fg(x) = f(g(x))$

Grade D — **Find the inverse of a function**
Write $y = f(x)$, rearrange to make x the subject, then write the result in terms of x

Grade C — **State the range of a function**
Use a sketch over the given domain; write the range in terms of $f(x)$, e.g. $f(x) \geq 2$

Grade C — **Sketch a function and its inverse**
The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$

Grade B — **State the domain and range of an inverse**
The domain of f^{-1} is the range of f; the range of f^{-1} is the domain of f

Grade B — **Solve equations involving composite functions**
Form $fg(x)$ algebraically, set it equal to the value and solve.

Grade A — **Restrict domains and find ranges of composites**
Restrict a many-to-one function to one side of its turning point so it has an inverse; find the range of fg by applying f to the range of g

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