Curve sketching and the modulus function

P2.7 · Pure mathematics · revision notes

Sketching polynomial graphs from their factorised form, the reciprocal graphs $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ with their asymptotes, and the modulus graph $y = |ax + b|$. You use points of intersection to count or find solutions of equations, solve modulus equations and inequalities, and work with proportional relationships and their graphs.

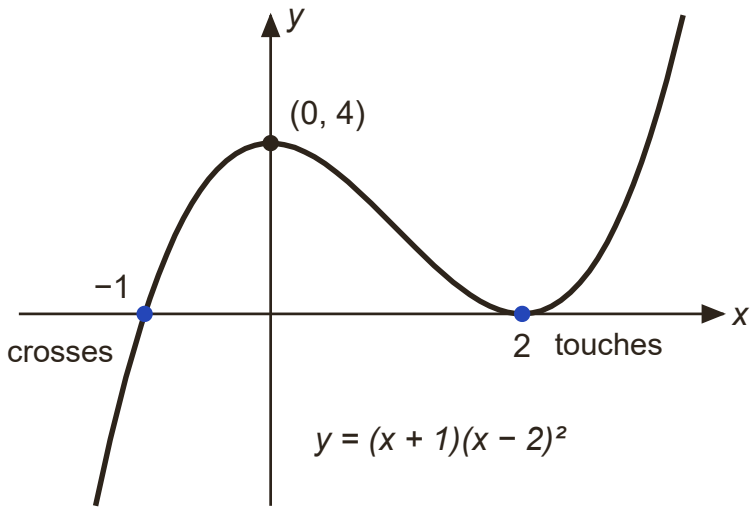
Key facts

- Root $x = a \Leftrightarrow$ factor $(x - a)$; y-intercept: put $x = 0$
- $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$: asymptotes $x = 0$ and $y = 0$
- $|x| = x$ if $x \geq 0$; $|x| = -x$ if $x < 0$
- Number of real solutions of $f(x) = g(x)$ = number of intersections
- Squared factor: the curve touches the x-axis there
- $y = \frac{a}{x^2}$, $a > 0$: always above the x-axis
- $|f(x)| = g(x)$: solve $f(x) = g(x)$ and $-f(x) = g(x)$, then check
- $y \propto x^n \Leftrightarrow y = kx^n$

Notes

Polynomial graphs

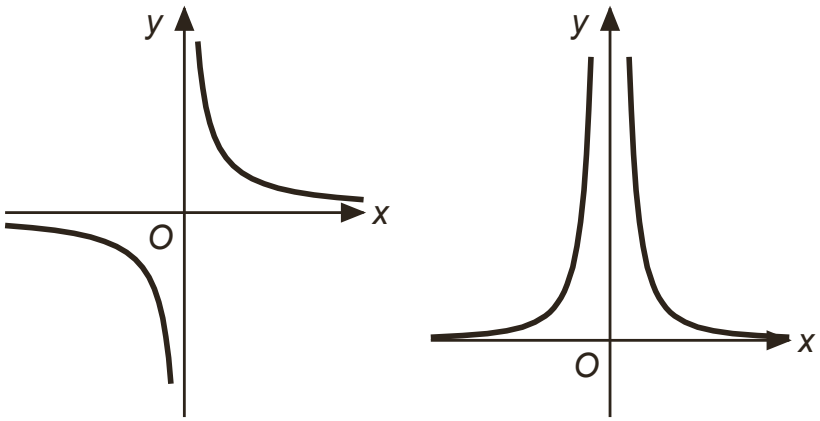
- Roots come from the factorised form (put $y = 0$); the y-intercept comes from putting $x = 0$
- Shape comes from the leading term: a positive cubic goes from bottom left to top right, a negative cubic from top left to bottom right. A quartic with positive x^4 coefficient has both ends going up.
- A repeated root $(x - a)^2$ touches the x-axis at $x = a$ (a turning point). A triple root $(x - a)^3$ flattens out and crosses.
- Example: $y = (x + 1)(x - 2)^2$ crosses at $x = -1$, touches at $x = 2$ and meets the y-axis at $(0, 4)$



A single factor gives a crossing point; a squared factor gives a point where the curve touches the axis.

Reciprocal graphs

- $y = \frac{a}{x}$ with $a > 0$: branches in the first and third quadrants. With $a < 0$: second and fourth quadrants.
- $y = \frac{a}{x^2}$ with $a > 0$: both branches above the x-axis, symmetric about the y-axis.



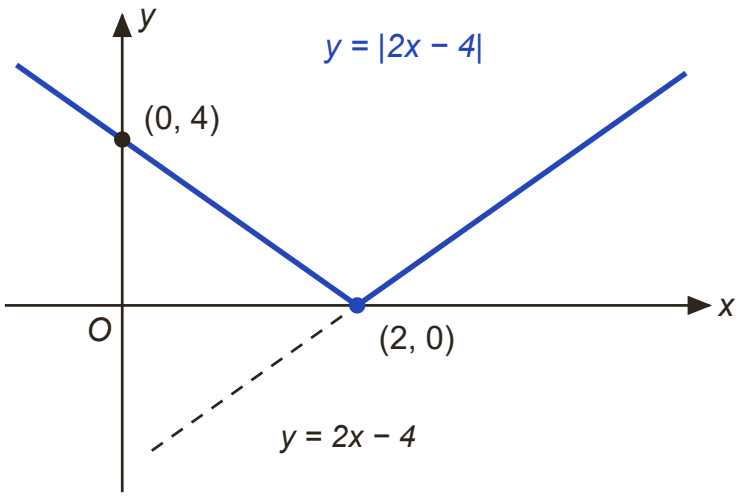
y = a/x (a > 0) y = a/x² (a > 0)

Both curves have the axes x = 0 and y = 0 as asymptotes. a/x² is never negative.

- Both have asymptotes $x = 0$ and $y = 0$: lines the curve gets closer and closer to but never reaches. State them as equations.
- Proportion: $y = kx$ is a straight line through the origin; $y = \frac{k}{x}$ is inverse proportion; $y \propto x^n$ means $y = kx^n$

The modulus function

- $|x|$ is the non-negative size of x: $|3| = 3$ and $|-3| = 3$
- To sketch $y = |ax + b|$, sketch $y = ax + b$ and reflect any part below the x-axis in the x-axis. The graph is a V shape with its vertex at $\left(-\frac{b}{a}, 0\right)$



The dashed part of y = 2x − 4 below the axis is reflected upwards to make the V.

- To solve $|ax + b| = cx + d$, solve $ax + b = cx + d$ and $-(ax + b) = cx + d$, then check each answer, because one may not satisfy the original equation.
- For modulus inequalities, find where the graphs meet, then read off from a sketch where one graph is above the other.

Intersections

- The solutions of $f(x) = g(x)$ are the x-coordinates of the points where $y = f(x)$ and $y = g(x)$ meet.
- So you can state the number of real solutions of an equation by counting intersections on a sketch.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

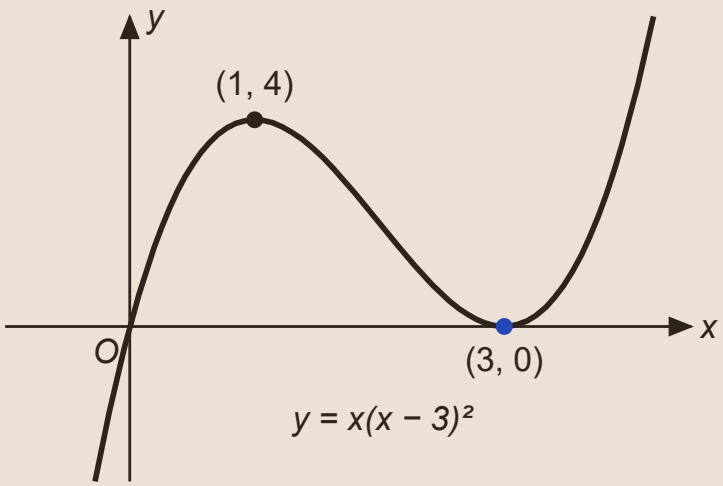
Sketch a polynomial, showing where it meets the axes

3 marks | Grade D

1. Find the roots from the factors and note any repeated root.
2. Find the y-intercept.
3. Use the leading term for the shape and label all intercepts with coordinates.

Example. Sketch the curve with equation $y = x(x-3)^2$, showing the coordinates of the points where it meets the coordinate axes.

Model answer.



Cubic shape rising from bottom left to top right (B1)

Passes through $(0, 0)$ (B1)

Touches the x-axis at $(3, 0)$, a minimum point (B1)

Solve a modulus equation

4 marks | Grade B

1. Solve the positive case: $ax + b = g(x)$
2. Solve the negative case: $-(ax + b) = g(x)$
3. Check each answer in the original equation (the right-hand side cannot be negative) and reject any that fail.

Example. Solve $|x - 5| = 2x - 1$

Model answer.

$-(x-5) = 2x - 1$, so $5 - x = 2x - 1$ (M1)

$x = 2$ (A1)

$x - 5 = 2x - 1$ gives $x = -4$ (M1)

At $x = -4$, $2x - 1 = -9$, but a modulus cannot be negative, so reject it: $x = 2$ only. (A1)

Solve a modulus inequality

4 marks **Grade B**

1. Find the x-coordinates where the graphs meet (both cases).
2. Sketch both graphs.
3. Read off where the inequality holds.

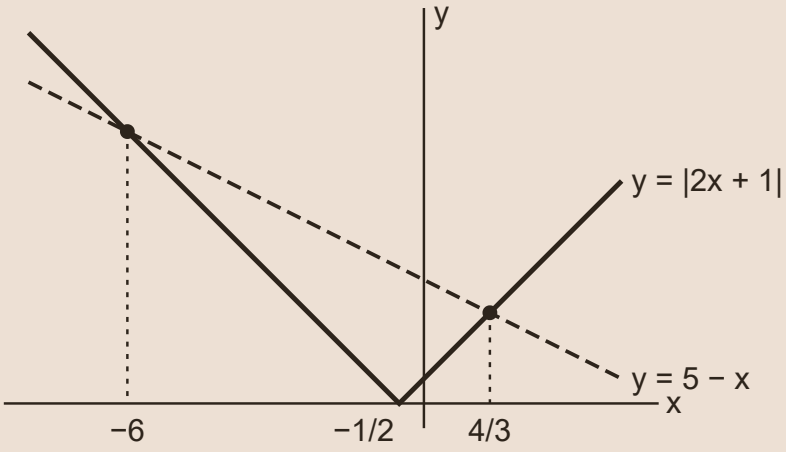
Example. Find the set of values of x for which $|2x + 1| > 5 - x$

Model answer.

$2x + 1 = 5 - x$ gives $x = \frac{4}{3}$ (M1)

$-(2x + 1) = 5 - x$ gives $x = -6$ (A1 both)

Sketch $y = |2x + 1|$ (a V with vertex $\left(-\frac{1}{2}, 0\right)$) and the line $y = 5 - x$ (dashed):



The V is above the line to the left of $x = -6$ and to the right of $x = \frac{4}{3}$ (M1)

$x < -6$ or $x > \frac{4}{3}$ (A1)

Use a sketch to find the number of solutions

4 marks | Grade C

1. Sketch both graphs on the same axes, with key points and asymptotes.
2. Count the points of intersection.
3. State the number of solutions with a reason.

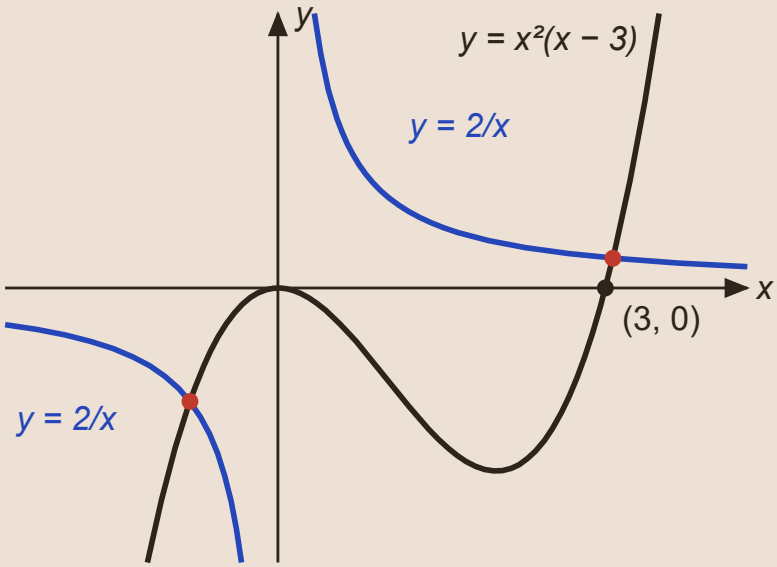
Example. (a) On the same axes, sketch the curves $y = x^2(x-3)$ and $y = \dfrac{2}{x}$.

(b) Hence state the number of real solutions of the equation $x^3(x-3) = 2$

Model answer.

(a) Cubic touching the origin and crossing the x-axis at $(3, 0)$ (B1)

Reciprocal curve in the first and third quadrants, with the axes as asymptotes (B1)



(b) $x^3(x-3) = 2$ is equivalent to $x^2(x-3) = \dfrac{2}{x}$ ($x \neq 0$), so count intersections (M1)

The curves meet once for $x > 3$ and once for $x < 0$, so there are 2 real solutions. (A1)

Don't lose marks

- Drawing a squared factor as a crossing point instead of a touching point.
- Letting a reciprocal curve cross, touch or curl away from its asymptote.
- Not rejecting invalid solutions of modulus equations.
- Drawing $y = \dfrac{a}{x^2}$ (with $a > 0$) with a branch below the x-axis.
- Not giving coordinates such as $(3, 0)$ and $(0, 0)$ when the question asks for them.

More tips

Memory tricks

- Sketch checklist: shape, x-intercepts, y-intercept, asymptotes labelled with equations.
- For $y = (x - a)(x - b)(x - c)$, the y-intercept is $(-a)(-b)(-c)$
- A modulus graph can never go below the x-axis.
- Check modulus solutions by substituting: the right-hand side must not be negative.

Exam technique

- 'Sketch' means show the key features (shape, intercepts, asymptotes), not plot points accurately; label everything the question asks for.
- After 'hence' with a sketch, count the intersections of the graphs you drew and give a reason.
- Asymptotes must be stated as equations, e.g. $x = 0$, not just drawn.
- A quick sketch before solving a modulus equation shows how many solutions to expect.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Sketch a cubic from its factorised form**
Mark the roots and the y-intercept; a positive x^3 coefficient goes from bottom left to top right.

Grade D — **Show repeated roots correctly on a sketch**
A squared factor such as $(x - 2)^2$ means the curve touches the x-axis at $x = 2$ without crossing.

Grade D — **Sketch reciprocal graphs and their asymptotes**
$y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ both have asymptotes $x = 0$ and $y = 0$

Grade D — **Use proportional relationships and their graphs**
$y \propto x^n$ means $y = kx^n$; find k from one pair of values.

Grade C — **Sketch the modulus graph of a linear function**
Reflect the part of $y = ax + b$ below the x-axis to make a V shape, vertex at $x = -\frac{b}{a}$

Grade C — **Use intersections to count solutions**
The number of points where $y = f(x)$ and $y = g(x)$ meet is the number of real solutions of $f(x) = g(x)$

Grade B — **Solve modulus equations and inequalities**
Solve both $ax + b = g(x)$ and $-(ax + b) = g(x)$, then check each answer with a sketch.

Grade A — **Find values of k for a number of intersections**
Use a sketch (and sometimes the discriminant) to find where a line or curve meets a graph a given number of times.

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