Polynomials, algebraic division and the factor theorem

P2.6 · Pure mathematics · revision notes

Working with polynomials: expanding, factorising, dividing by a linear expression and using the factor theorem to find the factors of a cubic. You also simplify algebraic fractions. A typical question says 'show that $(x-2)$ is a factor of f(x), hence factorise f(x) completely' for 4 to 6 marks, often followed by a sketch or by solving $f(x)=0$

Key facts

- Factor theorem: $f(a)=0 \Leftrightarrow (x-a)$ is a factor
- $f(-3)=0 \Leftrightarrow (x+3)$ is a factor
- $f(x) \equiv (x-a)(px^2+qx+r)$: compare coefficients
- $a^2-b^2=(a-b)(a+b)$
- $f\left(\frac{b}{a}\right)=0 \Leftrightarrow (ax-b)$ is a factor
- First factor: try $\pm$ the factors of the constant term
- Put in $0x^2$, $0x$ for missing terms when dividing
- Cancel common factors (brackets), never single terms

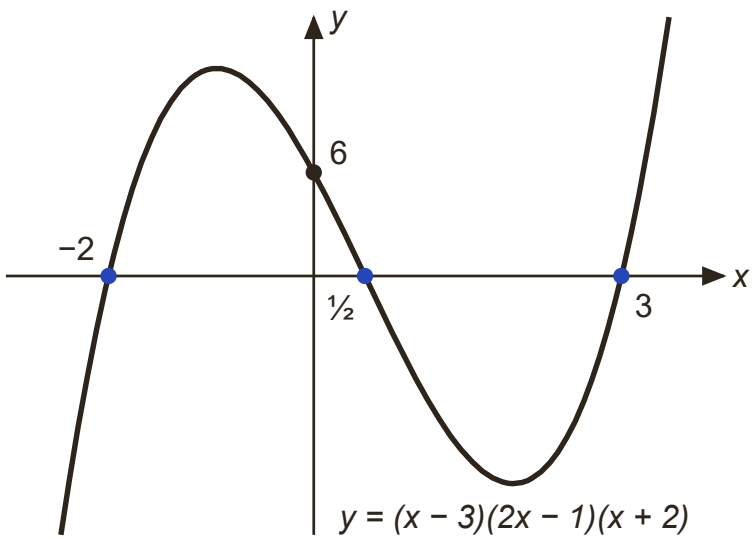
Notes

Polynomials

- A polynomial is a sum of terms ax^n, where n is a non-negative integer. Its degree is the highest power: a cubic has degree 3.
- Expand by multiplying each term in the first bracket by each term in the second, then collect like terms.
- Write $f(x)$ in descending powers of x, and include zero terms for any missing powers when you divide, e.g. x^3+0x^2-7x+6

Algebraic division

- To divide by $(x - a)$: divide the leading term, multiply back, subtract, bring down the next term, and repeat.
- Example: $(2x^3 - 3x^2 - 11x + 6) \div (x - 3)$ gives $2x^2 + 3x - 2$, with no remainder.
- Alternatively compare coefficients: write $f(x) \equiv (x - 3)(ax^2 + bx + c)$; the x^3 terms give a, the constants give c, then find b
- Then $2x^2 + 3x - 2 = (2x - 1)(x + 2)$, so $2x^3 - 3x^2 - 11x + 6 = (x - 3)(2x - 1)(x + 2)$



Each factor gives a root: (x + 2) → −2, (2x − 1) → ½, (x − 3) → 3. The constant term 6 is the y-intercept.

The factor theorem

- If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$. Conversely, if $(x - a)$ is a factor, then $f(a) = 0$
- To test $(x + 3)$, work out $f(-3)$. Be careful with the sign.
- $(ax - b)$ is a factor if and only if $f\left(\frac{b}{a}\right) = 0$; e.g. test $(2x - 1)$ with $f\left(\frac{1}{2}\right)$
- To find a first factor, try factors of the constant term: $\pm 1, \pm 2, \pm 3, \ldots$
- For $f(x) = x^3 - 7x + 6$, $f(1) = 0$, so $(x - 1)$ is a factor and $f(x) = (x - 1)(x^2 + x - 6)$
- Factorise the quadratic: $f(x) = (x - 1)(x + 3)(x - 2)$

Algebraic fractions

- Factorise the numerator and denominator fully, then cancel common factors. Cancel whole brackets, never individual terms.
- Example:

$$\begin{aligned}&\frac{x^2-16}{3x^2+11x-4}\\&=\frac{(x-4)(x+4)}{(3x-1)(x+4)}\\&=\frac{x-4}{3x-1}\end{aligned}$$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Show a factor, then factorise completely 6 marks **Grade C**

1. Substitute the value into $f(x)$, show it equals 0 and conclude.
2. Divide $f(x)$ by the factor (or compare coefficients) to get a quadratic.
3. Factorise the quadratic.
4. Write $f(x)$ as a product of three linear factors.

Example. $f(x)=2x^3+x^2-13x+6$
(a) Use the factor theorem to show that $(x-2)$ is a factor of $f(x)$.
(b) Hence factorise $f(x)$ completely.

Model answer.

(a) $f(2)=2(8)+4-26+6$ (M1)
$=0$, so $(x-2)$ is a factor. (A1)
(b) Dividing by $(x-2)$ or comparing coefficients (M1)
$f(x)=(x-2)(2x^2+5x-3)$ (A1)
Factorising the quadratic (M1)
$f(x)=(x-2)(2x-1)(x+3)$ (A1)

Simplify an algebraic fraction

3 marks | Grade C

1. Take out any common factor first.
2. Factorise the numerator and the denominator fully.
3. Cancel the common bracket.

Example. Simplify fully $\dfrac{x^3 - 4x}{x^2 + x - 6}$

Model answer.

$x^3 - 4x = x(x-2)(x+2)$ (M1)

$x^2 + x - 6 = (x+3)(x-2)$ (B1)

$\dfrac{x(x+2)}{x+3}$ (A1)

Find unknown coefficients using the factor theorem

6 marks | Grade A

1. For each factor $(ax - b)$, set $f\left(\dfrac{b}{a}\right) = 0$
2. Simplify each to a linear equation in the unknowns.
3. Solve the equations simultaneously.

Example. $f(x) = 2x^3 + ax^2 + bx - 6$, where a and b are constants. Given that $(x+1)$ and $(2x-3)$ are both factors of $f(x)$, find the values of a and b

Model answer.

$f(-1) = -2 + a - b - 6 = 0$ (M1)

$a - b = 8$ (A1)

$f\left(\dfrac{3}{2}\right)$

$= \dfrac{27}{4} + \dfrac{9}{4}a + \dfrac{3}{2}b - 6$

$= 0$ (M1)

$3a + 2b = -1$ (A1)

Solving simultaneously (M1)

$a = 3,\ b = -5$ (A1)

Don't lose marks

- Sign error in the factor theorem: to test $(x+2)$, substitute $x = -2$, not $x = 2$
- No conclusion: '$f(2) = 0$, so $(x-2)$ is a factor' earns the accuracy mark.

- Stopping at the quadratic: 'completely' means three linear factors when possible.
- Missing out zero terms in long division, so the columns do not line up.
- Cancelling terms instead of factors, e.g. cancelling the x^2 in $\dfrac{x^2+3}{x^2+5x}$

More tips

Memory tricks

- $f(1)$ is the sum of the coefficients: if they add up to 0, $(x-1)$ is a factor.
- For ax^3+bx^2+cx+d, $f(-1)=-a+b-c+d$: quick to check in your head.
- When comparing coefficients, the x^2 coefficient of the quotient is the leading coefficient of f divided by the coefficient of x in the divisor.
- Check a factorisation by expanding it, or by substituting $x=1$ into both forms.

Exam technique

- 'Use the factor theorem' means you must substitute and show the value is 0; dividing alone does not score that mark.
- Show the substitution with numbers, e.g. $f(2)=2(8)+4-26+6=0$, not just '$f(2)=0$'.
- 'Hence' means use the factor you have just shown, not a new method.
- The factorised form gives the roots and y-intercept you need for a sketch in the next part.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Expand brackets and collect like terms**
Multiply every term in one bracket by every term in the other, e.g. $(x+2)(x^2-3x+1)=x^3-x^2-5x+2$

Grade D — **Show a factor using the factor theorem**
Show $f(a)=0$ and conclude that $(x-a)$ is a factor of $f(x)$

Grade C — **Divide a cubic by a linear factor**
Use long division or compare coefficients to find the quadratic factor.

Grade C — **Factorise a cubic completely**
Find one factor with the factor theorem, divide, then factorise the quadratic quotient.

Grade C — **Simplify algebraic fractions by factorising**
Factorise the top and bottom fully, then cancel common factors (whole brackets only).

Grade B — **Use factors of the form ax − b**
$(ax-b)$ is a factor of $f(x)$ if and only if $f\left(\dfrac{b}{a}\right)=0$

Grade A — **Find unknown coefficients from given factors**
Each known factor gives an equation, e.g. $f(-1)=0$; solve the equations simultaneously.

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