Quadratics and the discriminant

P2.3 · Pure mathematics · revision notes

Quadratic functions and their graphs, completing the square, and the discriminant $b^2 - 4ac$, which tells you how many real roots a quadratic has. You also solve quadratics in a function of x, such as $x^4 - 10x^2 + 9 = 0$. Discriminant questions with an unknown constant k are very common and worth 4 to 6 marks.

Key facts

- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- > 0: two distinct real roots; $= 0$: repeated root; < 0: no real roots; ≥ 0: real roots
- Turning point of $y = a(x + p)^2 + q$ is $(-p, q)$
- $a > 0$: $\cup$ shape, minimum; $a < 0$: $\cap$ shape, maximum
- Discriminant: $b^2 - 4ac$
- $x^2 + bx = \left(x + \dfrac{b}{2}\right)^2 - \dfrac{b^2}{4}$
- Line of symmetry: $x = -\dfrac{b}{2a}$

Notes

Solving and the graph

- The roots of $ax^2 + bx + c = 0$ are where the graph of $y = ax^2 + bx + c$ meets the x-axis.
- Solve by factorising, by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, or by completing the square.
- $a > 0$: a $\cup$-shaped curve with a minimum point. $a < 0$: a $\cap$-shaped curve with a maximum point. The y-intercept is $(0, c)$

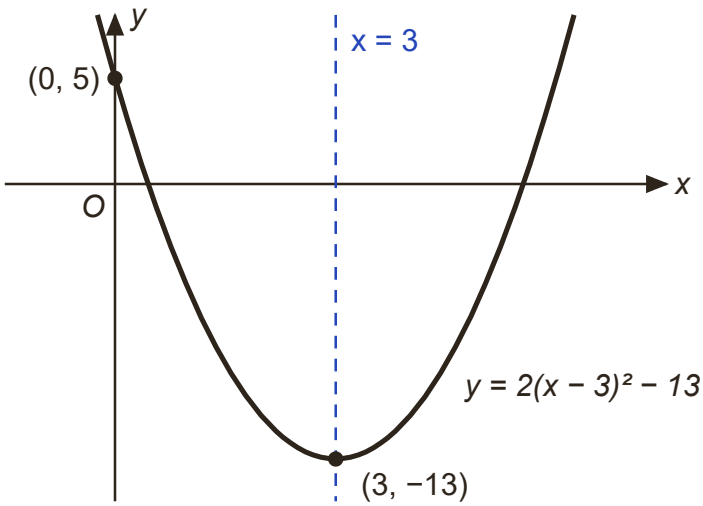
Completing the square

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

- With $a \neq 1$, take a out of the x terms first:

$$\begin{aligned} 2x^2 - 12x + 5 &= 2(x^2 - 6x) + 5 \\ &= 2(x - 3)^2 - 18 + 5 \\ &= 2(x - 3)^2 - 13 \end{aligned}$$

- $y = a(x + p)^2 + q$ has turning point $(-p, q)$ and line of symmetry $x = -p$

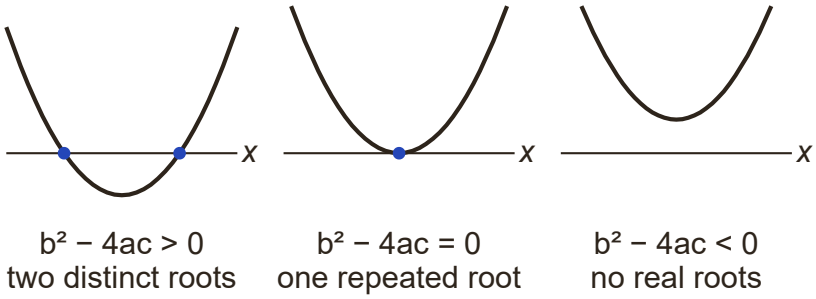


In completed-square form you can read off the turning point (3, −13) and the line of symmetry x = 3.

- If $a > 0$, the minimum value is q, so if $q > 0$ the expression is always positive.

The discriminant

- The discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$
- $b^2 - 4ac > 0$: two distinct real roots (the curve crosses the x-axis twice).
- $b^2 - 4ac = 0$: one repeated root (the curve touches the x-axis).
- $b^2 - 4ac < 0$: no real roots (the curve does not meet the x-axis).



The discriminant tells you how many times the curve meets the x-axis.

- 'Real roots' includes the repeated case, so it means $b^2 - 4ac \geq 0$
- With an unknown k: write down a, b and c in terms of k, form the equation or inequality, and solve it.

Quadratics in a function of x

- $x^4 - 10x^2 + 9 = 0$ is $u^2 - 10u + 9 = 0$ with $u = x^2$, so $u = 1$ or 9, giving $x = \pm 1$ or ± 3
- $x - 5\sqrt{x} + 6 = 0$ uses $u = \sqrt{x}$: $u = 2$ or 3, so $x = 4$ or 9
- Reject impossible values: $x^2 = -2$, $\sqrt{x} = -1$ and $2^x = -3$ have no real solutions.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, ft = follow through: you still earn the mark if you use your own earlier wrong answer correctly.

Complete the square and find the turning point 3 to 4 marks **Grade D**

1. Take out the coefficient of x^2 from the x terms.
2. Halve the coefficient of x inside the bracket and subtract its square (multiplied by a).
3. Read the turning point from $a(x + p)^2 + q$: it is $(-p, q)$

Example. (a) Write $3x^2 + 12x + 5$ in the form $a(x + b)^2 + c$.
(b) Hence state the coordinates of the minimum point of the curve $y = 3x^2 + 12x + 5$

Model answer.

(a) $3(x^2 + 4x) + 5$, so $a = 3$	(B1)
$3(x + 2)^2 - 12 + 5$	(M1)
$3(x + 2)^2 - 7$	(A1)
(b) $(-2, -7)$	(B1 ft)

Find the values of k using the discriminant

4 marks **Grade B**

1. Identify a, b and c in terms of k, using brackets.
2. Use the right condition: > 0, $= 0$, ≥ 0 or < 0
3. Simplify to the printed inequality if there is one.
4. Solve the quadratic inequality in k using a sketch.

Example. The equation $x^2 + (k-3)x + (3-2k) = 0$, where k is a constant, has two distinct real roots.
(a) Show that $k^2 + 2k - 3 > 0$.
(b) Find the set of possible values of k

Model answer.

(a) Two distinct real roots, so $b^2 - 4ac > 0$: $(k-3)^2 - 4(1)(3-2k) > 0$ (M1)
$k^2 - 6k + 9 - 12 + 8k > 0$, so $k^2 + 2k - 3 > 0$ (A1)
(b) $(k+3)(k-1) = 0$, critical values $k = -3$ and $k = 1$ (M1)
$k < -3$ or $k > 1$, i.e. $\{k : k < -3\} \cup \{k : k > 1\}$ (A1)

Solve a quadratic in disguise

4 marks **Grade B**

1. Spot the substitution, e.g. $2^{2x} = (2^x)^2$, so let $u = 2^x$
2. Solve the quadratic in u
3. Convert back to x and reject any impossible values.

Example. Solve $2^{2x} - 12 \times 2^x + 32 = 0$

Model answer.

Let $u = 2^x$: $u^2 - 12u + 32 = 0$ (M1)
$(u-4)(u-8) = 0$, so $u = 4$ or $u = 8$ (M1)
$2^x = 4$ gives $x = 2$ (A1)
$2^x = 8$ gives $x = 3$ (A1)

Use the discriminant to find a tangent

4 marks | Grade A

1. Set the line equal to the curve and form a quadratic equal to 0.
2. A tangent means one repeated root: $b^2 - 4ac = 0$
3. Solve for the unknown.

Example. The line $y = mx - 1$ is a tangent to the curve $y = x^2 + 3$. Find the possible values of m

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Model answer.

$x^2 + 3 = mx - 1$, so $x^2 - mx + 4 = 0$	(M1)
Tangent: $b^2 - 4ac = 0$	(M1)
$m^2 - 16 = 0$	(A1)
$m = 4$ or $m = -4$	(A1)

Don't lose marks

- Squaring b wrongly: $(k - 3)^2 = k^2 - 6k + 9$, not $k^2 - 9$
- Completing the square with $a \neq 1$ without taking a out first, or forgetting to multiply the subtracted square by a
- Wrong signs for the turning point: $3(x + 2)^2 - 7$ has turning point $(-2, -7)$, not $(2, -7)$
- Stopping at u in a disguised quadratic instead of finding x, or keeping an impossible value such as $2^x = -3$
- When k multiplies x^2, forgetting that $k = 0$ makes the equation linear, not quadratic.

More tips

Memory tricks

- Label a, b and c before using the discriminant, with brackets: $b = (k - 3)$
- Check a completed square by expanding it back in your head.
- The x-coordinate of the turning point is halfway between the roots.
- 'Real roots' includes equal roots, so use ≥ 0; 'distinct' means > 0

Exam technique

- For a printed inequality in k, your working must lead exactly to it, with the inequality sign kept all the way through.
- Two separate intervals need 'or' (or $\cup$), not 'and'. Give set notation if the question asks for it.
- Sketch the quadratic in k to decide whether you need the region between or outside the critical values.
- 'Hence' means use the previous part: the completed square gives the turning point directly.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E

Solve a quadratic by factorising or formula

Factorise when you can; otherwise use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Grade D

Complete the square

$x^2 + 6x + 2 = (x + 3)^2 - 7$; when $a \neq 1$, take a out of the x terms first.

Grade D

Sketch a quadratic with intercepts and vertex

Show the shape, the y-intercept, any roots and the turning point from the completed square.

Grade C

Use the discriminant to count real roots

$b^2 - 4ac > 0$: two distinct real roots; $= 0$: one repeated root; < 0: no real roots.

Grade B

Find the range of k for given roots

Form an inequality in k from the discriminant and solve it, often as a quadratic inequality.

Grade B

Solve a quadratic in a function of x

Put $u = x^2$, $u = \sqrt{x}$ or $u = 2^x$, solve for u, then find x

Grade A

Use the discriminant for tangency conditions

Substitute a line into a curve and use $b^2 - 4ac = 0$ for the line to be a tangent.

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