Partial fractions

P2.10 · Pure mathematics · revision notes

Splitting a fraction with a factorised denominator into a sum of simpler fractions. Denominators are products of linear factors, possibly including one squared factor, giving at most three partial fractions. Partial fractions are often part (a) of a longer question that then asks you to integrate, differentiate or expand using the binomial series.

Key facts

- $\dfrac{\dots}{(x-a)(x-b)} \equiv \dfrac{A}{x-a} + \dfrac{B}{x-b}$
- $\dfrac{\dots}{(x-a)(x-b)(x-c)} \equiv \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
- $\dfrac{\dots}{(x-a)(x-b)^2} \equiv \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{(x-b)^2}$
- Grade A or above only Improper (numerator degree $\geq$ denominator degree): include a polynomial part or divide first
- Compare coefficients (e.g. of x^2) for any constant still unknown
- Substitute the roots of the denominator to find constants quickly
- $\equiv$ means identically equal: true for every value of x

Notes

Distinct linear factors

- Write $\dfrac{5x+1}{(x-1)(x+2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+2}$
- Multiply both sides by $(x-1)(x+2)$: $5x+1 \equiv A(x+2) + B(x-1)$
- Substitute values that make a bracket zero: $x = 1$ gives $6 = 3A$, so $A = 2$; $x = -2$ gives $-9 = -3B$, so $B = 3$
- So $\dfrac{5x+1}{(x-1)(x+2)} \equiv \dfrac{2}{x-1} + \dfrac{3}{x+2}$. Three distinct factors work the same way with three fractions.
- Factorise the denominator first if necessary, e.g. $x^2 - 4 = (x-2)(x+2)$

Repeated linear factors

$$\frac{f(x)}{(x-a)(x-b)^2} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}$$

- Multiplying through: $f(x) \equiv A(x-b)^2 + B(x-a)(x-b) + C(x-a)$
- Substituting $x = a$ gives A and $x = b$ gives C. Find B by comparing coefficients of x^2, or by substituting another value such as $x = 0$

Improper fractions Grade A or above only

- A fraction is improper if the degree of the numerator is greater than or equal to the degree of the denominator.
- Divide first, or include a polynomial part in the identity: $\frac{2x^2 + 6x - 5}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$
- Comparing x^2 coefficients gives $A = 2$; substituting $x = 1$ and $x = -2$ gives $B = 1$ and $C = 3$

Why they are useful Grade A or above only

- Partial fractions turn one hard fraction into simple ones you can integrate, e.g. $\int \frac{3}{x+2}\,dx = 3\ln|x+2| + c$, or expand term by term with the binomial series.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Express in partial fractions: distinct linear factors

4 marks | Grade C

1. Write the correct form with one fraction per factor.
2. Multiply through by the denominator to get an identity.
3. Substitute each root to find each constant.
4. Write the final answer as a sum of fractions.

Example. Express $\dfrac{4x^2+5x-21}{(x-1)(x+2)(x-3)}$ in partial fractions.

Model answer.

$\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{C}{x-3}$ (B1)

$4x^2+5x-21 \equiv A(x+2)(x-3)$

$+B(x-1)(x-3)+C(x-1)(x+2)$

Substituting, e.g. $x=1$: $-12=-6A$ (M1)

Two of $A=2$, $B=-1$, $C=3$ (A1)

$\dfrac{2}{x-1}-\dfrac{1}{x+2}+\dfrac{3}{x-3}$ (A1)

Express in partial fractions: repeated factor

5 marks | Grade B

1. Include both $\dfrac{B}{x+1}$ and $\dfrac{C}{(x+1)^2}$ for the squared factor.
2. Substitute the roots to find two constants.
3. Compare coefficients of x^2 to find the last one.

Example. Express $\dfrac{3x^2+11x+6}{x(x+1)^2}$ in partial fractions.

Model answer.

$\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$ (B1)

$3x^2+11x+6 \equiv A(x+1)^2+Bx(x+1)+Cx$ and substituting (M1)

$x=0$: $A=6$; $x=-1$: $-2=-C$, so $C=2$ (A1)

x^2 coefficients: $A+B=3$ (M1)

$B=-3$, so $\dfrac{6}{x}-\dfrac{3}{x+1}+\dfrac{2}{(x+1)^2}$ (A1)

Improper fraction with a polynomial part

4 marks | Grade A

1. Spot that the fraction is improper.
2. Find the polynomial part by comparing leading coefficients (or dividing).
3. Find the other constants by substitution.

Example. Given that $\dfrac{3x^2-9}{(x+1)(x-2)} \equiv A + \dfrac{B}{x+1} + \dfrac{C}{x-2}$, find the values of the constants A, B and C

Model answer.

$A = 3$ (comparing x^2 coefficients) (B1)

$3x^2 - 9 \equiv 3(x+1)(x-2) + B(x-2) + C(x+1)$ and substituting (M1)

$x = -1$: $-6 = -3B$, so $B = 2$ (A1)

$x = 2$: $3 = 3C$, so $C = 1$ (A1)

Hence integrate using partial fractions

4 marks | Grade A

1. Use your partial fractions from the previous part.
2. Integrate each $\dfrac{k}{x-a}$ term as $k\ln|x-a|$
3. Integrate $\dfrac{k}{(x-a)^2}$ as $-\dfrac{k}{x-a}$, and add $+c$

Example. Using the result $\dfrac{3x^2+11x+6}{x(x+1)^2} \equiv \dfrac{6}{x} - \dfrac{3}{x+1} + \dfrac{2}{(x+1)^2}$, find

$$\int \frac{3x^2+11x+6}{x(x+1)^2}\,dx$$

Model answer.

$\ln$ terms from the first two fractions (M1)

$6\ln|x| - 3\ln|x+1|$ (A1)

$\int 2(x+1)^{-2}\,dx = -2(x+1)^{-1}$ (M1)

$6\ln|x| - 3\ln|x+1| - \dfrac{2}{x+1} + c$ (A1)

Don't lose marks

- Leaving out the $\dfrac{B}{x-b}$ term when the denominator has $(x-b)^2$
- Not noticing an improper fraction, so the polynomial part is missing.

- Arithmetic slips when substituting negative values, e.g. $x = -2$ into $(x - 3)$
- Forgetting to factorise the denominator first.
- Finding A, B and C but not writing the final answer as partial fractions.

More tips

Memory tricks

- Cover-up check: for a non-repeated factor $(x - a)$, cover it in the original fraction and substitute $x = a$ into what is left to get its constant.
- Check your answer: substitute $x = 0$ (or any easy value) into the original and into your partial fractions; they must agree.
- The number of unknown constants equals the degree of the denominator, plus the polynomial part if the fraction is improper.

Exam technique

- Give the final answer as a sum of fractions, unless the question only asks for the constants.
- Later parts (integrating or expanding) usually allow follow-through from your partial fractions, so always attempt them.
- If substitution gets awkward, compare coefficients and say which coefficient you are comparing.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Split a fraction with two linear factors**

$$\frac{px + q}{(x - a)(x - b)} \equiv \frac{A}{x - a} + \frac{B}{x - b}$$

Grade C — **Find the constants by substitution**

Multiply through by the denominator and substitute the roots, e.g. $x = a$, to find A and B quickly.

Grade C — **Handle three distinct linear factors**

Use $\frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c}$ and substitute each root in turn.

Grade B — **Handle a repeated linear factor**

For $(x - b)^2$ you need both $\frac{B}{x - b}$ and $\frac{C}{(x - b)^2}$

Grade B — **Compare coefficients to find a constant**

Equate the coefficients of x^2 (or the constants) when substitution does not give every constant.

Grade A — **Split an improper algebraic fraction**

If the numerator's degree is at least the denominator's, include a polynomial part, e.g.

$$A + \frac{B}{x - 1} + \frac{C}{x + 2}$$

Grade A — **Use partial fractions to integrate or expand**

Each simple fraction can be integrated (giving a $\ln$ term) or expanded with the binomial series.

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