Laws of indices

P2.1 · Pure mathematics · revision notes

The laws of indices for all rational powers, including zero, negative and fractional indices. They are tested directly in short 2 to 4 mark questions (often 'write in the form ax^n' or 'solve by writing both sides as powers of 2') and they are needed all the time in calculus and exponentials.

Key facts

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1 \, (a \neq 0)$
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
- $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$
- $(ab)^n = a^n b^n$

Notes

The laws

- $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$ and $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$ and $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- $a^0 = 1$ for any $a \neq 0$
- The first three laws only combine powers of the same base: $2^3 \times 3^2$ cannot be written as a single power.

Negative and fractional indices

- $a^{-n} = \dfrac{1}{a^n}$: a negative power means 'one over'. It does not make the answer negative.
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. Take the root first to keep numbers small: $32^{\frac{3}{5}} = 2^3 = 8$
- Deal with the minus sign, then the root, then the power:

$$\begin{aligned}&\left(\frac{4}{9}\right)^{-\frac{3}{2}}\\&= \left(\frac{9}{4}\right)^{\frac{3}{2}}\\&= \left(\frac{3}{2}\right)^{3}\\&= \frac{27}{8}\end{aligned}$$

Writing in the form ax^n

- $\sqrt{x} = x^{\frac{1}{2}}$, $x\sqrt{x} = x^{\frac{3}{2}}$, $\dfrac{1}{x^3} = x^{-3}$ and $\dfrac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$
- A number on the bottom stays as a number: $\dfrac{5}{x^2} = 5x^{-2}$, but $\dfrac{1}{5x^2} = \dfrac{1}{5}x^{-2}$
- Split a fraction with a single-term denominator: $\dfrac{x^3 - 6}{x^2} = x - 6x^{-2}$
- You cannot split a fraction whose denominator has two terms: $\dfrac{1}{x+2}$ is not $x^{-1} + 2^{-1}$
- Watch brackets: $(3x)^2 = 9x^2$, but $3x^2$ means $3 \times x^2$

Equations with indices

- Write both sides as powers of the same base, then equate the powers: $9^x = 27^{x-1}$ gives $3^{2x} = 3^{3x-3}$, so $x = 3$
- To undo a power $\dfrac{m}{n}$, raise both sides to the power $\dfrac{n}{m}$: $x^{\frac{3}{2}} = 64$ gives $x = 64^{\frac{2}{3}} = 16$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Evaluate a power without a calculator 2 marks **Grade D**

1. If the power is negative, flip the fraction.
2. Take the root given by the denominator of the power.
3. Raise to the power given by the numerator.

Example. Find the exact value of $\left(\frac{8}{125}\right)^{-\frac{2}{3}}$

Model answer.

$\left(\frac{125}{8}\right)^{\frac{2}{3}}$, or the cube root $\frac{5}{2}$ or $\frac{2}{5}$ seen (M1)

$\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ (A1)

Write an expression as a sum of powers of x 2 to 3 marks **Grade C**

1. Write any roots as fractional powers.
2. Divide every term on the top by the bottom.
3. Use $a^m \div a^n = a^{m-n}$ and keep the numbers as exact fractions.

Example. Given that $y = \frac{4x^3 - 3\sqrt{x}}{2x}$, write y in the form $px^a + qx^b$, where p, q, a and b are constants.

Model answer.

$\sqrt{x} = x^{\frac{1}{2}}$ and dividing each term by $2x$ (M1)

$\frac{4x^3}{2x} = 2x^2$ (A1)

$\frac{3x^{\frac{1}{2}}}{2x} = \frac{3}{2}x^{-\frac{1}{2}}$, so $y = 2x^2 - \frac{3}{2}x^{-\frac{1}{2}}$ (A1)

Write as a single power of a number

2 marks | Grade C

1. Write every number as a power of the same base (e.g. $9 = 3^2$, $\sqrt{3} = 3^{\frac{1}{2}}$).
2. Combine the powers with the index laws.

Example. Express $\dfrac{9^x}{\sqrt{3}}$ in the form 3^y, giving y in terms of x

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Model answer.

$9^x = 3^{2x}$ or $\sqrt{3} = 3^{\frac{1}{2}}$ (M1)

$\dfrac{3^{2x}}{3^{\frac{1}{2}}} = 3^{2x-\frac{1}{2}}$, so $y = 2x - \dfrac{1}{2}$ (A1)

Solve an equation by using a common base

4 marks | Grade B

1. Write both sides as powers of the same prime.
2. Simplify each side to a single power.
3. Equate the powers and solve the linear equation.

Example. In this question you must show all stages of your working.

Solve $\dfrac{2^{3x}}{4^{x-1}} = 8\sqrt{2}$

Model answer.

$4^{x-1} = 2^{2x-2}$, so the left side is $2^{3x-(2x-2)} = 2^{x+2}$ (M1)

$8\sqrt{2} = 2^3 \times 2^{\frac{1}{2}}$

$= 2^{\frac{7}{2}}$ (B1)

$x + 2 = \dfrac{7}{2}$ (M1)

$x = \dfrac{3}{2}$ (A1)

Don't lose marks

- Thinking a negative power gives a negative number: $2^{-2} = \dfrac{1}{4}$, not -4
- Multiplying the bases: $2^3 \times 2^4 = 2^7$, not 4^7
- Moving the coefficient into the power: $\dfrac{3}{x^2} = 3x^{-2}$, not $(3x)^{-2}$
- Not dividing every term when splitting a fraction: $\dfrac{x^2+1}{x} = x + x^{-1}$

✗ $(x^3)^2 = x^6$, not x^5 or x^9

More tips

Memory tricks

- Negative power: flip. Fractional power: the bottom is the root, the top is the power.
- Know powers of 2 up to $2^{10} = 1024$, powers of 3 up to $3^5 = 243$, and $4^3 = 64$, $5^3 = 125$, so you can spot common bases.
- Check a simplification by putting a value such as $x = 4$ into both forms on your calculator.
- Recognise $x\sqrt{x} = x^{\frac{3}{2}}$ and $\dfrac{1}{x\sqrt{x}} = x^{-\frac{3}{2}}$: they come up constantly in calculus.

Exam technique

- All papers allow a calculator, but index questions often say 'show all stages of your working': write each step, e.g. for $\left(\dfrac{27}{8}\right)^{\frac{2}{3}}$ write $\left(\dfrac{27}{8}\right)^{\frac{1}{3}} = \dfrac{3}{2}$ before squaring.
- Give a and n in 'the form ax^n' as exact numbers (fractions, not rounded decimals).
- Rewriting in powers of x is often part (a) of a calculus question; check it carefully because part (b) depends on it.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Use the multiplication, division and power laws**
$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$ and $(a^m)^n = a^{mn}$

Grade E — **Evaluate zero and negative powers**
$a^0 = 1$ and $a^{-n} = \dfrac{1}{a^n}$, e.g. $2^{-3} = \dfrac{1}{8}$

Grade D — **Evaluate fractional powers without a calculator**
$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$, e.g. $27^{\frac{2}{3}} = 3^2 = 9$

Grade C — **Write expressions as powers of x**
e.g. $\dfrac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$, ready to differentiate or integrate.

Grade C — **Split a fraction into separate powers**
e.g. $\dfrac{x^2 + 4}{2\sqrt{x}} = \dfrac{1}{2}x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}$

Grade B — **Solve equations using a common base**
e.g. $8^x = 4^{x+1}$ gives $2^{3x} = 2^{2x+2}$, so $x = 2$

Grade B — **Solve equations with fractional or negative powers**
e.g. $x^{-\frac{3}{2}} = \dfrac{1}{8}$ gives $x^{\frac{3}{2}} = 8$, so $x = 8^{\frac{2}{3}} = 4$

Grade A — **Simplify expressions with several different bases**
e.g.

$$\begin{aligned}\frac{2^{x+1}\times 4^{x}}{8^{x-1}} &= 2^{x+1+2x-3x+3}\\ &= 2^{4}\\ &= 16\end{aligned}$$

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