Vector problems

P10.5 · Pure mathematics · revision notes

Vectors let you solve problems about shapes and about real situations such as forces and motion. In Pure you prove properties of quadrilaterals and other shapes; in context you find resultant forces, use equilibrium and describe the positions of moving objects with $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$. These are multi-step questions worth 5 to 10 marks, closely linked to Mechanics.

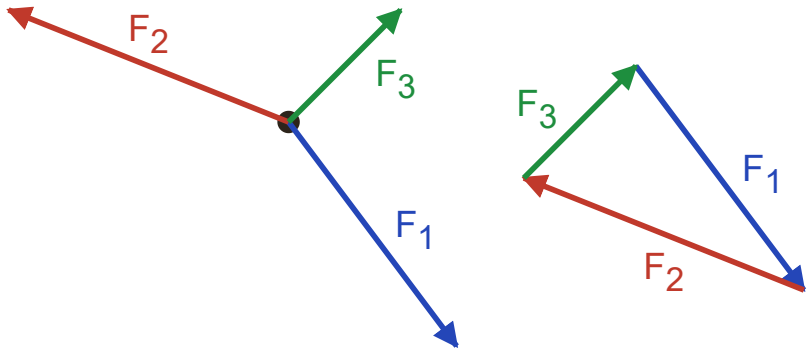
Key facts

- Resultant: $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots$
- Constant velocity: $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
- Vector from A to B: $\mathbf{r}_B - \mathbf{r}_A$
- $\mathbf{F} = m\mathbf{a}$
- Equilibrium: $\sum \mathbf{F} = \mathbf{0}$
- Speed $= |\mathbf{v}|$
- Parallelogram: $\overrightarrow{AB} = \overrightarrow{DC}$; rhombus: also $|\overrightarrow{AB}| = |\overrightarrow{AD}|$

Notes

Forces

- The resultant of several forces is their vector sum. Find its magnitude by Pythagoras and its direction by trigonometry.
- A particle is in equilibrium when the resultant force is zero: $\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = \mathbf{0}$



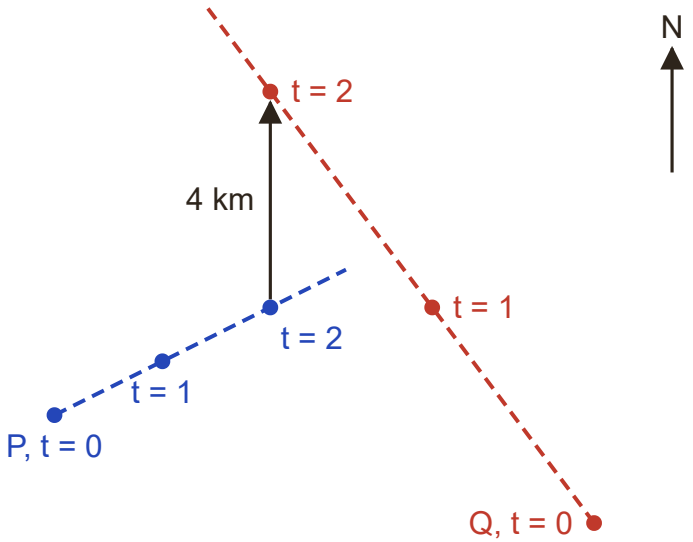
forces on the particle — nose to tail: a closed triangle

In equilibrium the forces add to $\mathbf{0}$: drawn nose to tail they close up. Here $\mathbf{F_3} = 2\mathbf{i} + 2\mathbf{j}$

- Newton's second law works with vectors too: $\mathbf{F} = m\mathbf{a}$

Motion with constant velocity

- Velocity $\mathbf{v}$ is a vector; speed is its magnitude, $|\mathbf{v}|$
- Position at time t: $\mathbf{r} = \mathbf{r_0} + \mathbf{v}t$. Write it as one vector with $\mathbf{i}$ and $\mathbf{j}$ components in terms of t
- The vector from ship A to ship B is $\mathbf{r}_B - \mathbf{r}_A$; the distance between them is its magnitude.
- With $\mathbf{i}$ east and $\mathbf{j}$ north: 'B is due north of A' means equal $\mathbf{i}$-components, with B having the larger $\mathbf{j}$-component; 'B is north-east of A' means $\overrightarrow{AB}$ has equal, positive components.



At $t = 2$ both ships have $\mathbf{i}$-component 1, so Q is due north of P, 4 km away.

Proving properties of shapes

- Parallelogram: one pair of opposite sides are equal vectors (e.g. $\overrightarrow{AB} = \overrightarrow{DC}$).
- Rhombus: a parallelogram with two adjacent sides of equal length.
- Trapezium: one pair of opposite sides parallel (one a scalar multiple of the other) but of different lengths.
- Kite: two pairs of adjacent sides of equal length.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Use equilibrium to find a force

4 marks | Grade C

1. Write the sum of all the forces equal to $\mathbf{0}$
2. Add the known forces and make the unknown force the subject.
3. Find its magnitude with Pythagoras, giving units.

Example. A particle is in equilibrium under three forces, $\mathbf{F}_1 = (3\mathbf{i} - 4\mathbf{j})$ N, $\mathbf{F}_2 = (-5\mathbf{i} + 2\mathbf{j})$ N and $\mathbf{F}_3$.
Find $\mathbf{F}_3$ and its magnitude.

Model answer.

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$ **M1**

$\mathbf{F}_3 = -(-2\mathbf{i} - 2\mathbf{j})$
$= (2\mathbf{i} + 2\mathbf{j})$ N **A1**

$|\mathbf{F}_3| = \sqrt{2^2 + 2^2}$ **M1** $= 2\sqrt{2}$ N (about 2.83 N) **A1**

Use position vectors of moving objects

6 marks | Grade A

1. Write each position vector in terms of t using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
2. Translate the condition (e.g. 'due north') into an equation between components.
3. Solve for t, then answer in context (a clock time, a distance).

Example. [$\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north, and distances are in km.]
At noon, ship P is at $(-3\mathbf{i} + 5\mathbf{j})$ km and moves with constant velocity $(2\mathbf{i} + \mathbf{j})$ km h^{-1}. At the same time, ship Q is at $(7\mathbf{i} + 3\mathbf{j})$ km and moves with constant velocity $(-3\mathbf{i} + 4\mathbf{j})$ km h^{-1}.
(a) Find the speed of Q.
(b) Find the time at which Q is due north of P, and the distance between the ships at that time.

Model answer.

(a) $|-3\mathbf{i} + 4\mathbf{j}| = \sqrt{9 + 16}$ **M1** $= 5$ km h^{-1} **A1**

(b) After t hours, $\mathbf{r}_P = (-3 + 2t)\mathbf{i} + (5 + t)\mathbf{j}$ and $\mathbf{r}_Q = (7 - 3t)\mathbf{i} + (3 + 4t)\mathbf{j}$ **M1**

Due north: $-3 + 2t = 7 - 3t$ **M1**, so $t = 2$, i.e. at 2 pm **A1**

At $t = 2$: $\mathbf{r}_P = \mathbf{i} + 7\mathbf{j}$ and $\mathbf{r}_Q = \mathbf{i} + 11\mathbf{j}$, so the ships are 4 km apart **A1**

Prove that a quadrilateral is a rhombus 4 marks **Grade B**

1. Show that one pair of opposite sides are equal vectors, so the shape is a parallelogram.
2. Show that two adjacent sides have equal lengths.
3. Conclude with the property that makes it a rhombus.

Example. The points $A(1, 1, 2)$, $B(4, 5, 2)$, $C(4, 8, 6)$ and $D(1, 4, 6)$ are the vertices of the quadrilateral $ABCD$.
Show that $ABCD$ is a rhombus.

Model answer.

$\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{DC} = 3\mathbf{i} + 4\mathbf{j}$	**M1**
$\overrightarrow{AB} = \overrightarrow{DC}$, so $ABCD$ is a parallelogram	**A1**
$\overrightarrow{AD} = 3\mathbf{j} + 4\mathbf{k}$; $\lvert\overrightarrow{AB}\rvert = \sqrt{9+16} = 5$ and $\lvert\overrightarrow{AD}\rvert = \sqrt{9+16} = 5$	**M1**
A parallelogram with two adjacent sides equal is a rhombus	**A1***

Don't lose marks

- Adding the starting position and the velocity without multiplying the velocity by t
- Giving a velocity when the question asks for a speed (a magnitude), or leaving out units.
- Proving a parallelogram but stopping before checking adjacent sides for a rhombus.
- Treating t as a clock time: $t = 2$ hours after noon is 2 pm.

More tips

Memory tricks

- Equilibrium: add the known forces; the missing force is the negative of that sum.
- Write positions as a single vector in t: $(x_0 + v_x t)\mathbf{i} + (y_0 + v_y t)\mathbf{j}$
- Compass words to components: due north or south means equal $\mathbf{i}$-components; due east or west means equal $\mathbf{j}$-components.
- 3-4-5 and 5-12-13 triangles appear often in speeds and forces.

Exam technique

- Read the bracketed information at the start of the question: it gives the directions of $\mathbf{i}$ and $\mathbf{j}$ and the units.
- Finish with a conclusion in context, e.g. 'at 2 pm the ships are 4 km apart'.
- In proofs, state which property you have shown (equal vectors, equal lengths) and what it means for the shape.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D

Find a resultant force

Add the force vectors: $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots$

Grade C

Use equilibrium: resultant force is zero

Set the sum of all the forces equal to $\mathbf{0}$ and solve for the unknown force.

Grade C

Find position with constant velocity

$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, where $\mathbf{r}_0$ is the position when $t = 0$

Grade B

Find speed and direction from velocity

Speed is $|\mathbf{v}|$; the direction is an angle or bearing found with trigonometry.

Grade B

Prove the type of a quadrilateral

Compare the sides as vectors (parallel, equal) and as lengths.

Grade A

Solve for unknown times or constants

For example, 'due north' means the $\mathbf{i}$-components of the two position vectors are equal.

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