Position vectors and distances

P10.4 · Pure mathematics · revision notes

The position vector of a point A is $\overrightarrow{OA} = \mathbf{a}$, its vector from the origin. The vector from A to B is $\mathbf{b} - \mathbf{a}$, and the distance between the points is its magnitude. You use this to find distances, midpoints, points that divide a line segment in a ratio, missing vertices of shapes, and side lengths and angles of triangles.

Key facts

- $\overrightarrow{OA} = \mathbf{a}$ (position vector of A)
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
- Distance $AB = |\mathbf{b} - \mathbf{a}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- 3D distance: add $(z_2 - z_1)^2$ under the square root
- Midpoint: $\frac{1}{2}(\mathbf{a} + \mathbf{b})$
- $AP : PB = m : n$: $\overrightarrow{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a})$
- Parallelogram $ABCD$: $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$
- Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

Notes

Position vectors

- Relative to a fixed origin O, the position vector of $A(x, y)$ is $\mathbf{a} = \overrightarrow{OA} = x\mathbf{i} + y\mathbf{j}$
- In 3D, $A(x, y, z)$ has position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

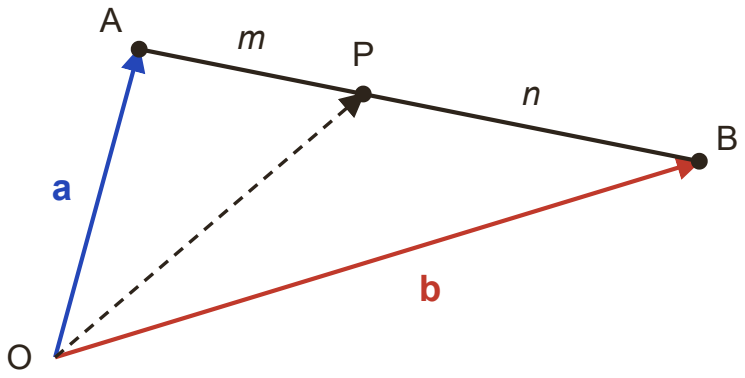
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= \mathbf{b} - \mathbf{a}\end{aligned}$$

Distances

- The distance AB is $|\mathbf{b} - \mathbf{a}|$. For $A(1, 2)$ and $B(4, 6)$: $\mathbf{b} - \mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, so $AB = \sqrt{9 + 16} = 5$
- In 3D, for $A(1, 2, 3)$ and $B(2, 4, 5)$: $\mathbf{b} - \mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, so $AB = \sqrt{1 + 4 + 4} = 3$
- Leave distances as exact surds unless asked for a decimal.

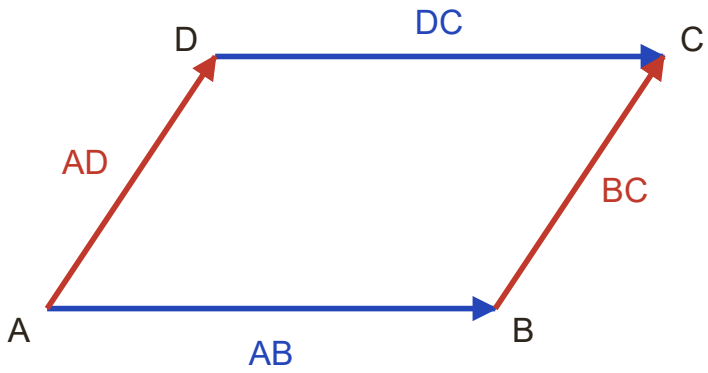
Special points

- Midpoint: $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, which is the average of the coordinates.
- P divides AB in the ratio $m : n$: start at A and go $\frac{m}{m+n}$ of the way to B



$\overrightarrow{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a})$: start at A and go $\frac{m}{m+n}$ of the way to B

- Parallelogram $ABCD$ (vertices in order): $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$



In parallelogram $ABCD$, opposite sides are equal vectors: $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

Triangles

- Find the three side lengths with the distance formula. Two equal sides show the triangle is isosceles; $a^2 + b^2 = c^2$ shows a right angle.
- Find an angle with the cosine rule, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, and the area with $\frac{1}{2}ab\sin C$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the distance between two points

3 marks **Grade C**

1. Find the vector between the points: end minus start.
2. Square the components, add and take the square root.

Example. The points A and B have coordinates $(2, -1)$ and $(-4, 7)$. Find $\overrightarrow{AB}$ and the distance AB

Model answer.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -6\mathbf{i} + 8\mathbf{j}$ **M1 A1**

$AB = \sqrt{36 + 64} = 10$ **A1**

Find a point that divides a line segment

3 marks **Grade C**

1. Find $\overrightarrow{AB}$
2. Turn the ratio into a fraction of the whole segment: $1 : 2$ means $\frac{1}{3}$ of the way.
3. Add that fraction of $\overrightarrow{AB}$ to $\mathbf{a}$

Example. A and B have position vectors $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $7\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$. The point P lies on AB with $AP : PB = 1 : 2$.
Find the position vector of P

Model answer.

$\overrightarrow{AB} = 6\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$ **M1**

$$\begin{aligned}\overrightarrow{OP} &= \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \\ &= (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &\quad + (2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) \quad \textbf{M1} \\ &= 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \textbf{A1}\end{aligned}$$

Find a missing vertex of a parallelogram

2 marks | Grade B

1. Use the order of the letters: in $ABCD$, $\overrightarrow{AD} = \overrightarrow{BC}$
2. Add the known side vector to the position vector of the starting vertex.

Example. $ABCD$ is a parallelogram, with its vertices in that order. A, B and C have coordinates $(1, 2)$, $(4, 3)$ and $(6, 0)$.
Find the coordinates of D

Model answer.

$\overrightarrow{AD} = \overrightarrow{BC} = 2\mathbf{i} - 3\mathbf{j}$, so $\mathbf{d} = \mathbf{a} + \overrightarrow{BC}$ **M1**

D is $(3, -1)$ **A1**

Find an angle in a triangle

4 marks | Grade A

1. Find the three vectors along the sides of the triangle.
2. Find the three side lengths.
3. Use the cosine rule for the angle required (the side opposite it goes after the minus sign).

Example. The points P, Q and R have coordinates $(1, 0, 2)$, $(3, 4, 1)$ and $(-1, 2, 5)$.
Find the size of angle QPR, giving your answer to 1 decimal place.

Model answer.

$\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{QR} = -4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ **M1**

$PQ = \sqrt{21}$, $PR = \sqrt{17}$, $QR = 6$ **A1**

$\cos QPR = \dfrac{21 + 17 - 36}{2\sqrt{21}\sqrt{17}}$ **M1**

Angle $QPR = 87.0°$ **A1**

Don't lose marks

- Finding $\mathbf{a} - \mathbf{b}$ for $\overrightarrow{AB}$: the distance is the same, but the vector points the wrong way.
- Writing the midpoint as $\frac{1}{2}(\mathbf{b} - \mathbf{a})$: that is half of $\overrightarrow{AB}$, not a position vector.
- Going $\frac{1}{2}$ of the way for a ratio $1 : 2$ instead of $\frac{1}{3}$
- Squaring negative differences wrongly in the distance formula.

More tips

Memory tricks

- 'End minus start' for any vector between two points.
- Midpoint: average each coordinate.
- Parallelogram check: after finding D, confirm that $\overrightarrow{DC} = \overrightarrow{AB}$
- The order of the letters matters: $ABCD$ means A to B to C to D round the shape.

Exam technique

- Show the vector between the points before its magnitude: it is usually a method mark.
- Give coordinates when asked for a point, and a vector when asked for a position vector.
- The scalar (dot) product is not part of A level Mathematics: angles come from side lengths and the cosine rule.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Write the position vector of a point**

The point $(3, -1)$ has position vector $3\mathbf{i} - \mathbf{j}$

Grade D — **Find the vector between two points**

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$: subtract the start from the end.

Grade C — **Find the distance between two points**

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Grade C — **Find a midpoint or dividing point**

$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$; for $AP : PB = m : n$, $\overrightarrow{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a})$

Grade B — **Find a missing vertex of a parallelogram**

Use equal opposite sides: in $ABCD$, $\overrightarrow{AD} = \overrightarrow{BC}$, so $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$

Grade A — **Find angles in a triangle from coordinates**

Find all three side lengths, then use the cosine rule.

Grade A — **Find unknown coordinates from a distance**

Form an equation from the distance squared and solve it, usually a quadratic.

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