Vector arithmetic and geometry

P10.3 · Pure mathematics · revision notes

Vectors can be added with the triangle law and multiplied by scalars, and both operations have geometric meanings. You use them to write routes around shapes in terms of given vectors, show that lines are parallel, prove that points are collinear and find the ratios in which lines divide each other. These are usually 4 to 7 mark proof-style questions.

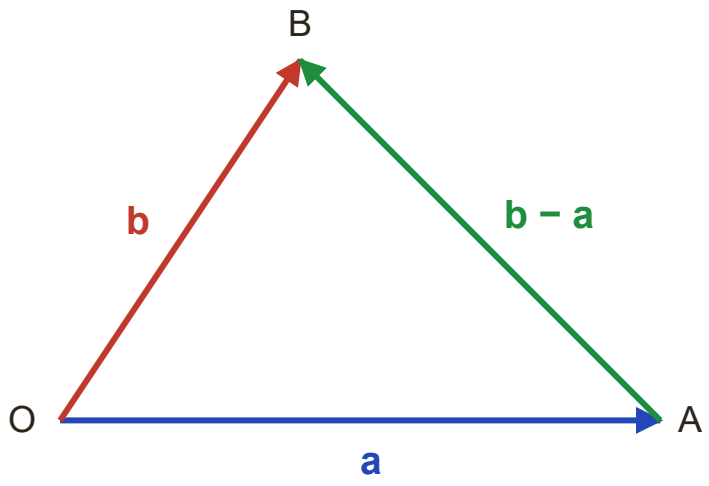
Key facts

- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\overrightarrow{BA} = -\overrightarrow{AB}$
- Midpoint of AB: $\frac{1}{2}(\mathbf{a} + \mathbf{b})$
- $AP : PB = m : n \Rightarrow \overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$
- Collinear: parallel vectors with a common point
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
- Parallel: $\mathbf{u} = \lambda\mathbf{v}$
- $\mathbf{a}$, $\mathbf{b}$ not parallel: $\lambda\mathbf{a} + \mu\mathbf{b} = p\mathbf{a} + q\mathbf{b} \Rightarrow \lambda = p$ and $\mu = q$

Notes

Adding and subtracting

- Triangle law: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. In a parallelogram, the diagonal from a corner is the sum of the two sides from that corner.
- Reversing a vector changes its sign: $\overrightarrow{BA} = -\overrightarrow{AB}$
- Any route from A to B gives the same vector. With $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$



Going from A to B via O: $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

Scalar multiples and parallel vectors

- $\lambda\mathbf{a}$ is parallel to $\mathbf{a}$, with magnitude $|\lambda||\mathbf{a}|$; for $\lambda < 0$ it points the opposite way.
- Two vectors are parallel exactly when one is a scalar multiple of the other. The scalar also gives the ratio of their lengths.

Ratios and midpoints

- If M is the midpoint of AB, $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$, so $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$
- If $AP : PB = 2 : 1$, P is $\frac{2}{3}$ of the way from A to B: $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$

Collinear points and intersections

- A, B and C are collinear if, for example, $\overrightarrow{AC} = k\overrightarrow{AB}$: the vectors are parallel **and** share the point A
- If $\mathbf{a}$ and $\mathbf{b}$ are non-zero and not parallel, and $\lambda\mathbf{a} + \mu\mathbf{b} = p\mathbf{a} + q\mathbf{b}$, then $\lambda = p$ and $\mu = q$
- To find where two lines meet, write the position vector of the meeting point in two ways (one unknown scalar for each line) and compare coefficients.

How to answer each type of question

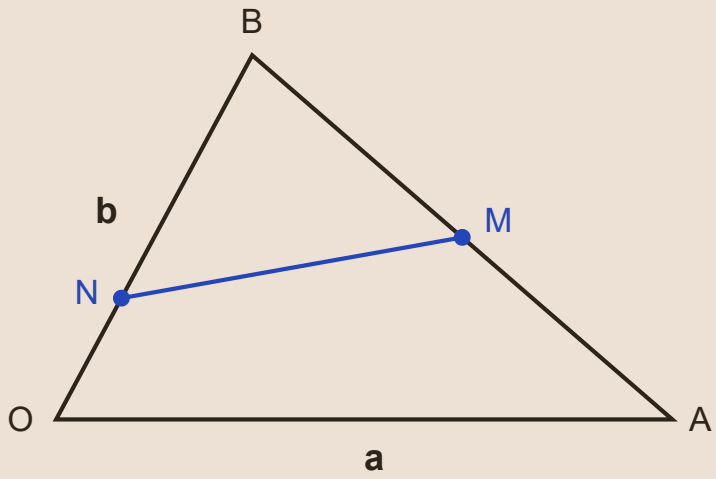
M = method mark, A = answer mark, B = a mark on its own.

Express vectors in terms of a and b 4 marks **Grade D**

1. Mark the given vectors and ratios on the diagram.
2. Choose a route made of vectors you know, and add them.
3. Simplify by collecting the $\mathbf{a}$ and $\mathbf{b}$ terms.

Example. In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB, and N is the point on OB with $ON : NB = 1 : 2$.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, (a) $\overrightarrow{OM}$, (b) $\overrightarrow{MN}$.



Model answer.

(a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, so $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$ **M1** $= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ **A1**

(b) $\overrightarrow{ON} = \frac{1}{3}\mathbf{b}$, so $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{3}\mathbf{b} - \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}$ **M1** $= -\frac{1}{2}\mathbf{a} - \frac{1}{6}\mathbf{b}$ **A1**

Prove that three points are collinear

3 marks **Grade B**

1. Find two vectors that join the points, e.g. $\overrightarrow{AB}$ and $\overrightarrow{AC}$
2. Show that one is a scalar multiple of the other.
3. State that they are parallel and share a point, so the points are collinear.

Example. Relative to an origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a} - 2\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel.
Prove that A, B and C are collinear, and find the ratio $AB : AC$

Model answer.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{AC} = 3\mathbf{a} - 2\mathbf{b} - \mathbf{a} = 2\mathbf{a} - 2\mathbf{b}$ **M1**

$\overrightarrow{AC} = -2\overrightarrow{AB}$, so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel **A1**

They share the point A, so A, B and C are collinear, and $AB : AC = 1 : 2$ (with A between B and C) **A1**

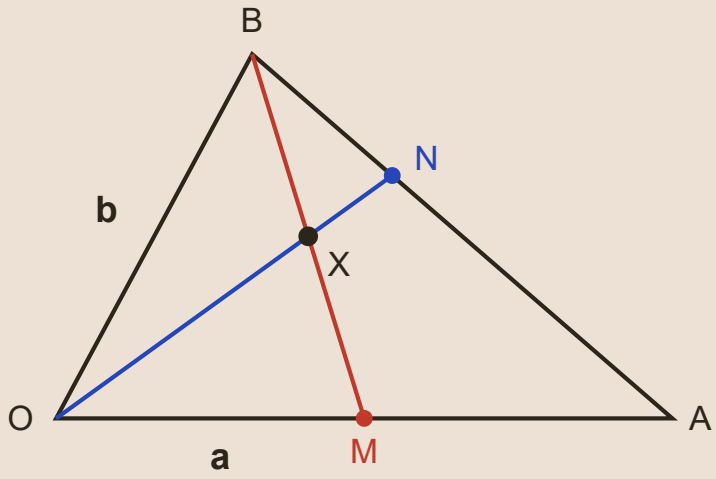
Find where two lines intersect

6 marks **Grade A**

1. Write the position vector of the meeting point using a scalar λ along one line.
2. Write it again using a scalar μ along the other line.
3. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, equate the coefficients of each and solve for λ and μ

Example. In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA, and N is the point on AB with $AN : NB = 2 : 1$. The lines ON and BM meet at X.

Find $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and the ratio $OX : XN$.



Model answer.

$$\overrightarrow{ON} = \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$$
$$= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} \quad \textbf{B1}$$

X on ON: $\overrightarrow{OX} = \lambda\left(\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$ **M1**

X on BM: $\overrightarrow{OX} = \mathbf{b} + \mu\left(\frac{1}{2}\mathbf{a} - \mathbf{b}\right)$

$$= \frac{1}{2}\mu\mathbf{a} + (1-\mu)\mathbf{b} \quad \textbf{M1}$$

$\mathbf{a}$ and $\mathbf{b}$ are not parallel, so $\frac{\lambda}{3} = \frac{\mu}{2}$ and $\frac{2\lambda}{3} = 1 - \mu$ **M1**

$\lambda = \frac{3}{4}$ and $\mu = \frac{1}{2}$ **A1**

$\overrightarrow{OX} = \frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}$, and $OX : XN = 3 : 1$ **A1**

Don’t lose marks

- Writing $\overrightarrow{AB} = \mathbf{a} - \mathbf{b}$
- Reading $AN : NB = 2 : 1$ as $\overrightarrow{AN} = \frac{1}{2}\overrightarrow{AB}$ instead of $\frac{2}{3}\overrightarrow{AB}$
- Concluding 'collinear' from parallel vectors alone, without mentioning the common point.
- Comparing coefficients without stating that $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

More tips

Memory tricks

- 'End minus start': $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
- Go round the shape: add each step, and change the sign when you travel against an arrow.
- Mark ratios on the diagram as fractions of the whole line before writing any vectors.
- Collinear = parallel + a shared point. Say both.

Exam technique

- Finish a proof with a clear statement, e.g. 'so A, B and C are collinear'.
- Simplify each vector fully (collect the $\mathbf{a}$ and $\mathbf{b}$ terms) so scalar multiples are easy to spot.
- Use a different letter for each unknown scalar (λ, μ) when writing two routes.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E

Add vectors using the triangle law

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$: follow the route from start to finish.

Grade D

Express a route in terms of given vectors

For example $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$

Grade C

Show that two vectors are parallel

Show that one is a scalar multiple of the other, e.g. $4\mathbf{a} - 6\mathbf{b} = 2(2\mathbf{a} - 3\mathbf{b})$

Grade C

Use a ratio to locate a point

If $AP : PB = m : n$, then $\overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$

Grade B

Prove that three points are collinear

Show that two of the vectors joining them are parallel and share a point.

Grade A

Find where lines meet by comparing routes

Write the same vector in two ways with unknown scalars, and compare the coefficients of two non-parallel vectors.

Free notes, Full packs, practice papers and mock exams at chhetriacademy.com. Written for this site; not taken from past papers. Stuck? 1-to-1 tutoring with a free 30-minute trial lesson.