# Magnitude and direction

P10.2 · Pure mathematics · revision notes

The magnitude of a vector is its length, found by Pythagoras in 2D or 3D, and its direction can be given by an angle with an axis or by a unit vector. You convert between component form and magnitude–direction form, which is used heavily in Mechanics for forces and velocities. Questions are short, typically 2 to 4 marks.

## Key facts

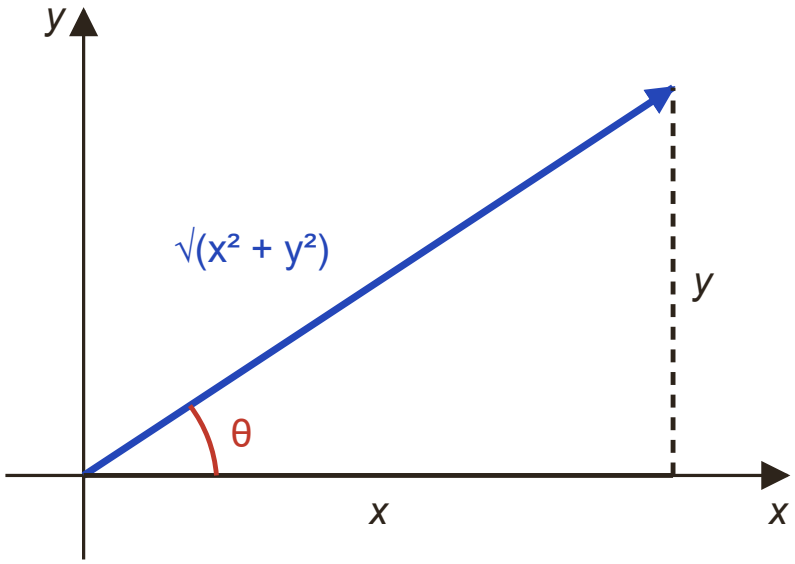
- $|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$
- $|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$
- Unit vector: $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$
- 2D direction: $\tan\theta = \dfrac{y}{x}$ (check the quadrant with a sketch)
- 3D: $\cos\theta_x = \dfrac{x}{|\mathbf{a}|}$, $\cos\theta_y = \dfrac{y}{|\mathbf{a}|}$, $\cos\theta_z = \dfrac{z}{|\mathbf{a}|}$
- Magnitude $m$ in the direction of $\mathbf{a}$: $m\hat{\mathbf{a}}$
- Magnitude $r$, angle $\theta$ from $\mathbf{i}$: $r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$

## Notes

### Magnitude

$$|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$$

- In 3D: $|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$




The magnitude is the hypotenuse, $\sqrt{x^2 + y^2}$, and $\tan\theta = \frac{y}{x}$

- Squares of negative components are positive:

$$|2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}| = \sqrt{4 + 36 + 9}$$
$$= 7$$

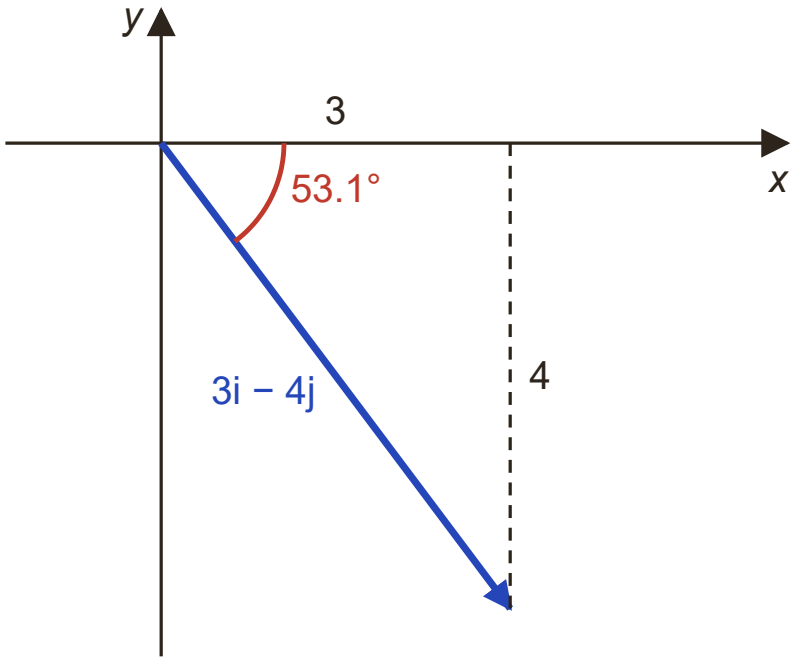
- A magnitude is never negative. In Mechanics, the magnitude of a velocity is the speed.

### Unit vectors

- A unit vector has magnitude 1. The unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$
- A vector of magnitude $m$ in the direction of $\mathbf{a}$ is $m\hat{\mathbf{a}}$; in the opposite direction it is $-m\hat{\mathbf{a}}$

## Direction in 2D

- The angle $\theta$ between $x\mathbf{i} + y\mathbf{j}$ and the positive $x$-axis satisfies $\tan\theta = \dfrac{y}{x}$. Sketch the vector first: the calculator's $\tan^{-1}$ only gives angles between $-90°$ and $90°$




Sketch first: $3\mathbf{i} - 4\mathbf{j}$ points into the fourth quadrant, $53.1°$ below the positive x-axis ($\tan\alpha = \dfrac{4}{3}$).

- From magnitude $r$ and angle $\theta$ measured anticlockwise from $\mathbf{i}$: $r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$
- Bearings (Mechanics) are measured clockwise from north, usually the $\mathbf{j}$ direction.

## Direction in 3D

- The angle $\theta_x$ between $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the positive $x$-axis satisfies $\cos\theta_x = \dfrac{x}{|\mathbf{a}|}$; similarly $\cos\theta_y = \dfrac{y}{|\mathbf{a}|}$ and $\cos\theta_z = \dfrac{z}{|\mathbf{a}|}$
- These angles are between $0°$ and $180°$: a negative component gives an obtuse angle.

## How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, ft = follow through: you still earn the mark if you use your own earlier wrong answer correctly.

### Find a magnitude and a unit vector

3 marks | Grade C

1. Square each component, add, and take the square root.
2. Divide every component by the magnitude.

**Example.** $\mathbf{a} = 5\mathbf{i} - 12\mathbf{j}$.
Find $|\mathbf{a}|$ and the unit vector in the direction of $\mathbf{a}$

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**Model answer.**

$$|\mathbf{a}| = \sqrt{5^2 + (-12)^2}$$
$$= \sqrt{169} \quad \textbf{M1} \quad = 13 \quad \textbf{A1}$$

$$\hat{\mathbf{a}} = \frac{1}{13}(5\mathbf{i} - 12\mathbf{j}) \quad \textbf{B1ft}$$

### Convert between components and magnitude–direction form

4 marks | Grade B

1. Components: multiply the magnitude by $\cos$ and $\sin$ of the angle from $\mathbf{i}$
2. Direction: sketch the vector, find an acute angle with $\tan^{-1}$, then describe it relative to the axis.

**Example.** (a) A force of magnitude 10 N acts at an angle of $150°$ anticlockwise from $\mathbf{i}$. Write the force in the form $(a\mathbf{i} + b\mathbf{j})$ N, giving exact values.
(b) Find the angle between the vector $3\mathbf{i} - 4\mathbf{j}$ and the positive $x$-axis.

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**Model answer.**

(a) $10\cos 150°\,\mathbf{i} + 10\sin 150°\,\mathbf{j}$ **M1** $= (-5\sqrt{3}\,\mathbf{i} + 5\mathbf{j})$ N **A1**

(b) The vector points into the fourth quadrant; $\tan\alpha = \frac{4}{3}$ **M1**

$\alpha = 53.1°$ below the positive $x$-axis (clockwise), equivalent to $306.9°$ anticlockwise **A1**

### Find the angle between a vector and an axis

3 marks **Grade A**

1. Find the magnitude of the vector.
2. Divide the component along that axis by the magnitude to get the cosine of the angle.
3. Use $\cos^{-1}$; a negative component gives an obtuse angle.

**Example.** Find the angle between the vector $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and the positive $y$-axis.

**Model answer.**

$|\mathbf{b}| = \sqrt{4 + 36 + 9} = 7$ **M1**

$\cos\theta_y = \dfrac{-6}{7}$ **M1**

$\theta_y = 149.0^\circ$ (1 d.p.) **A1**

### Find an unknown component from a magnitude

3 marks **Grade A**

1. Write the sum of the squared components equal to the magnitude squared.
2. Solve, keeping both square roots unless the question rules one out.

**Example.** The vector $\mathbf{v} = p\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, where $p$ is a constant, has magnitude 6. Find the possible values of $p$

**Model answer.**

$p^2 + 2^2 + (-4)^2 = 6^2$ **M1**

$p^2 = 16$ **A1**

$p = 4$ or $p = -4$ **A1**

## Don't lose marks

- Writing $(-4)^2$ as $-16$
- Giving the calculator's angle without checking which quadrant the vector is in.
- Dividing only some of the components by the magnitude when finding a unit vector.
- Using $\tan$ for the angle with an axis in 3D: use $\cos\theta = \dfrac{\text{component}}{\text{magnitude}}$
- Giving only the positive value when an unknown is squared.

## More tips

### Memory tricks

- Know some whole-number magnitudes: $(1, 2, 2)$ gives 3, $(2, 3, 6)$ gives 7, $(1, 4, 8)$ and $(4, 4, 7)$ give 9, $(2, 6, 9)$ gives 11.
- Check a unit vector: the squares of its components add up to 1.
- Always sketch a 2D vector before finding its angle.

### Exam technique

- State clearly which angle you have found, e.g. 'below the positive $x$-axis' or 'with the positive $y$-axis'.
- Leave magnitudes as exact surds (e.g. $\sqrt{29}$) unless a decimal is asked for.
- In Mechanics, the magnitude of a velocity is a speed and needs units, e.g. m $s^{-1}$.

## What each grade needs

What you need to be able to do, from the first marks up to the top grade.

**Grade E** — **Find the magnitude of a 2D vector**

$$|3\mathbf{i} - 4\mathbf{j}| = \sqrt{3^2 + (-4)^2} = 5$$

**Grade D** — **Find the magnitude of a 3D vector**

$$|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$$

**Grade C** — **Find a unit vector in a given direction**

Divide the vector by its magnitude: $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

**Grade C** — **Convert magnitude and angle to components**

A vector of magnitude $r$ at angle $\theta$ anticlockwise from $\mathbf{i}$ is $r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$

**Grade B** — **Find the direction of a 2D vector**

Use $\tan$ with a sketch to find the angle with the positive $x$-axis in the correct quadrant.

**Grade A** — **Find the angle between a vector and an axis**

In 3D, $\cos\theta_x = \dfrac{x}{|\mathbf{a}|}$, and similarly for the $y$- and $z$-axes.

**Grade A** — **Find unknowns from a given magnitude**

Set the sum of the squared components equal to the magnitude squared and solve, keeping both roots unless one is ruled out.

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