Vectors in 2D and 3D

P10.1 · Pure mathematics · revision notes

A vector has magnitude and direction. In A level Mathematics you write vectors as column vectors or with the unit vectors $\mathbf{i}$, $\mathbf{j}$ and, in three dimensions, $\mathbf{k}$, and you add, subtract and scale them component by component. Vectors appear in Pure (Papers 1 and 2) and throughout Mechanics (Paper 3), usually as the first part of a longer question.

Key facts

- $a\mathbf{i} + b\mathbf{j} = \begin{pmatrix} a \\ b \end{pmatrix}$
- $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
- $\mathbf{i}$, $\mathbf{j}$: unit vectors along the x- and y-axes
- $\mathbf{k}$: unit vector along the z-axis
- Add or subtract component by component
- $\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$
- Parallel: $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ
- Equal vectors: all components equal
- Point (x, y, z): position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Notes

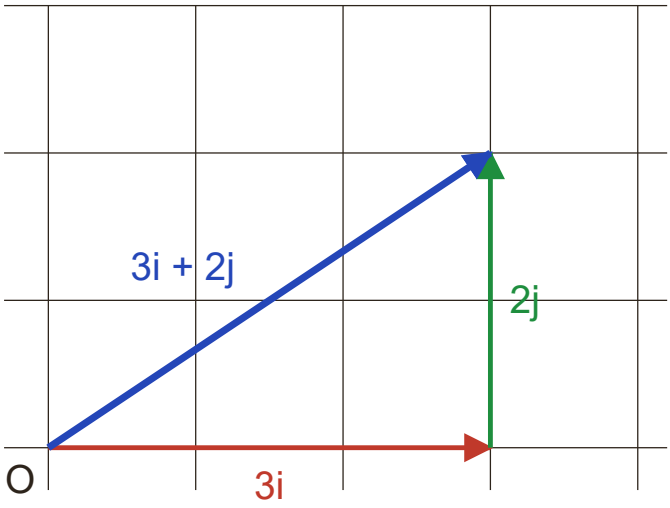
What a vector is

- A **scalar** has size only (e.g. mass, speed). A **vector** has size and direction (e.g. displacement, velocity, force).
- Notation: $\mathbf{a}$ in print, $\underline{a}$ (underlined) in handwriting, or $\overrightarrow{AB}$ for the vector from A to B
- $\mathbf{i}$ and $\mathbf{j}$ are unit vectors (magnitude 1) in the directions of the x- and y-axes.
- In 3D, $\mathbf{k}$ is the unit vector in the direction of the z-axis.

Component form

$$a\mathbf{i} + b\mathbf{j} = \begin{pmatrix} a \\ b \end{pmatrix}$$

- In 3D, $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$



$3\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$: 3 units in the $\mathbf{i}$ direction, then 2 in the $\mathbf{j}$ direction.

- In 3D the three axes are perpendicular to each other. The point (x, y, z) has position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- A missing component is zero: $3\mathbf{i} - \mathbf{k} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

Arithmetic

- Add or subtract matching components. To multiply by a scalar, multiply every component.
- $-\mathbf{a}$ has the same magnitude as $\mathbf{a}$ but the opposite direction. $\mathbf{0}$ is the zero vector.
- $\lambda\mathbf{a}$ is parallel to $\mathbf{a}$; if $\lambda < 0$ it points the opposite way.

Equating components

- $\mathbf{i}$ and $\mathbf{j}$ are independent: neither is a multiple of the other. So if $p\mathbf{i} + q\mathbf{j} = 5\mathbf{i} - 2\mathbf{j}$, then $p = 5$ and $q = -2$
- This turns one vector equation into simultaneous equations.
- In 3D, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are independent, giving three equations: with two unknowns, use the spare one as a check.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Combine vectors in component form

2 marks | Grade E

1. Multiply each vector by its scalar, component by component.
2. Add or subtract matching components, using brackets for negatives.

Example. $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 4\mathbf{j}$.
Find $3\mathbf{a} - 2\mathbf{b}$

Model answer.

$3\mathbf{a} - 2\mathbf{b} = (6\mathbf{i} - 3\mathbf{j}) - (-2\mathbf{i} + 8\mathbf{j})$ **M1**

$= 8\mathbf{i} - 11\mathbf{j}$ **A1**

Find scalars by equating components

3 marks | Grade C

1. Collect the $\mathbf{i}$ and $\mathbf{j}$ components on the left-hand side.
2. Equate each component with the right-hand side to get simultaneous equations.
3. Solve the simultaneous equations.

Example. Find the values of the constants p and q such that $p(2\mathbf{i} + \mathbf{j}) + q(\mathbf{i} - 3\mathbf{j}) = 7\mathbf{i} + 7\mathbf{j}$

Model answer.

$\mathbf{i}$: $2p + q = 7$; $\mathbf{j}$: $p - 3q = 7$ **M1**

$q = 7 - 2p \Rightarrow p - 21 + 6p = 7 \Rightarrow p = 4$ **M1**

$p = 4$, $q = -1$ **A1**

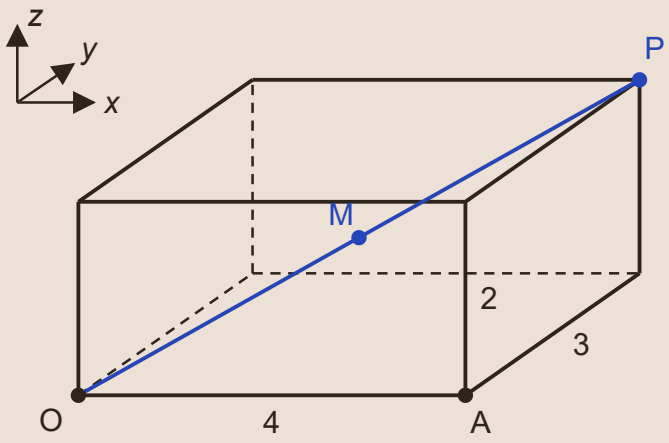
Use vectors to describe a cuboid

3 marks **Grade B**

1. Write down the coordinates of the vertices you need from the lengths of the edges.
2. Vector from one vertex to another = end minus start.
3. A midpoint is halfway along: halve the vector from the origin, or average the coordinates.

Example. A cuboid has one vertex at the origin O, and edges of lengths 4, 3 and 2 units along the positive x-, y- and z-axes. P is the vertex opposite O, A is the vertex $(4, 0, 0)$ and M is the midpoint of OP.

Find (a) $\overrightarrow{OP}$, (b) $\overrightarrow{AP}$, (c) the position vector of M.



Model answer.

(a) $\overrightarrow{OP} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ **B1**

(b) $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 3\mathbf{j} + 2\mathbf{k}$ **B1**

(c) $\frac{1}{2}\overrightarrow{OP} = 2\mathbf{i} + 1.5\mathbf{j} + \mathbf{k}$ **B1**

Don't lose marks

- Sign errors when subtracting a vector with negative components: use brackets.
- Forgetting that a missing component is zero, e.g. treating $3\mathbf{i} - \mathbf{k}$ as a 2D vector.
- Mixing up a point and a vector: the point $(3, -2, 5)$ has position vector $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$
- Not underlining vectors in handwritten answers, so $\mathbf{a}$ and a look the same.

More tips

Memory tricks

- Write vectors as columns when calculating: the components line up and the signs are easier to track.
- Parallel test: divide matching components; if all the ratios are the same, the vectors are parallel.
- Use the spare equation: with three components and two unknowns, check your answer in the third.

Exam technique

- Give answers in the same form as the question ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ or column vectors).
- Show the components you are equating: each equation is usually a method mark.
- In mechanics questions, check the units and which directions $\mathbf{i}$ and $\mathbf{j}$ represent.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E **Write vectors in i, j, k and column form**

$$3\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Grade E **Add, subtract and scale vectors**

Work component by component, e.g. $2(\mathbf{i} - 3\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) = 6\mathbf{i} - 5\mathbf{j}$

Grade D **Use 3D coordinates and axes**

The point $(2, -1, 4)$ has position vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ relative to the origin.

Grade C **Find unknowns by equating components**

If two vectors are equal, each pair of components is equal, giving equations to solve.

Grade C **Recognise parallel vectors**

Parallel vectors are scalar multiples of each other: every component is multiplied by the same number.

Grade B **Describe edges and diagonals of cuboids**

Write edges and diagonals of a cuboid in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ from the coordinates of its vertices.

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