Proof by contradiction

P1.1b · Pure mathematics · revision notes

In proof by contradiction you assume the statement is false and show that this leads to something impossible. You must know the proofs that $\sqrt{2}$ is irrational and that there are infinitely many primes, and be able to use the same method on unfamiliar statements. Expect 4 to 6 marks on Paper 1 or 2.

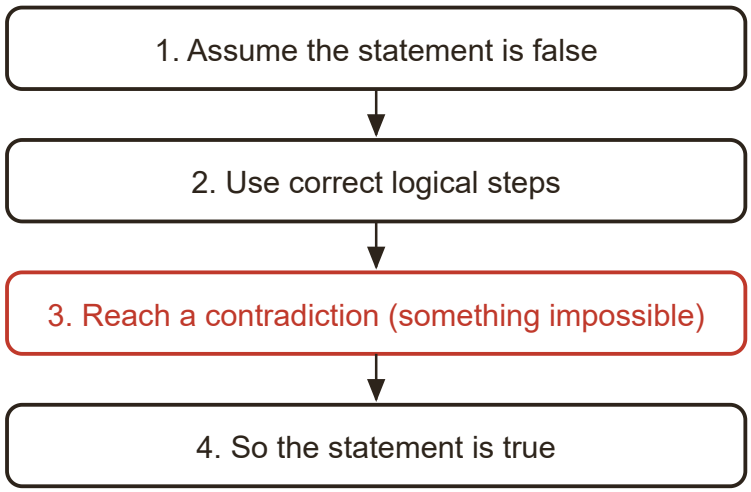
Key facts

- Assume the opposite $\longrightarrow$ logical steps $\longrightarrow$ contradiction $\longrightarrow$ the statement is true
- n^2 even $\Rightarrow$ n even; n^2 odd $\Rightarrow$ n odd
- Negation of 'for all x, P is true' is 'there is at least one x for which P is false'
- Rational $+$ irrational is irrational; non-zero rational $\times$ irrational is irrational
- Rational: $\frac{a}{b}$, a, b integers, $b \neq 0$, no common factors
- Negation of 'infinitely many' is 'finitely many'
- Primes proof: $N = p_1 p_2 \cdots p_n + 1$ leaves remainder 1 when divided by each p_i

Notes

The method

- Assume that the statement you want to prove is false (assume its negation).
- Use correct logical steps until you reach a contradiction: something impossible, such as a number that is both odd and even, or a fraction in lowest terms whose top and bottom share a factor.
- Conclude: 'This is a contradiction, so the assumption is false and the original statement is true.'



Every proof by contradiction has these four stages, and the last line must say what has been proved.

- Negations: 'there are infinitely many' becomes 'there are finitely many'; 'x is irrational' becomes 'x is rational, so $x = \frac{a}{b}$'; 'for all x' becomes 'there is at least one x for which it fails'.

The square root of 2 is irrational

- Assume $\sqrt{2} = \frac{a}{b}$, where a and b are integers with no common factors and $b \neq 0$
- Squaring: $a^2 = 2b^2$, so a^2 is even, so a is even. Write $a = 2c$
- Then $4c^2 = 2b^2$, so $b^2 = 2c^2$: b^2 is even, so b is even.
- So a and b have a common factor of 2, which contradicts the assumption. Hence $\sqrt{2}$ is irrational.
- The fact 'a^2 even $\Rightarrow$ a even' can itself be proved: if a were odd, a^2 would be odd.

There are infinitely many primes

- Assume there are finitely many primes, $p_1, p_2, \ldots, p_n$, and let $N = p_1 p_2 \cdots p_n + 1$
- Dividing N by any p_i leaves remainder 1, so no prime in the list is a factor of N
- But $N > 1$, so N is either prime or has a prime factor. Either way there is a prime not in the list: a contradiction.
- N itself need not be prime, for example:

$$\begin{aligned} 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 &= 30031 \\ &= 59 \times 509 \end{aligned}$$

Other contradiction proofs

- Integer equations: factorise, then use the fact that the factors are integers and look at odd and even.
- Irrationality: rational numbers are closed under $+$, $-$, $\times$ and $\div$ (by non-zero), so if $3+\sqrt{2}$ were rational then $\sqrt{2}$ would be too.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Prove a result about integers by contradiction 3 to 4 marks **Grade B**

1. Write the assumption: the statement is false.
2. Write the numbers algebraically (e.g. $n=2k$).
3. Show this leads to something impossible.
4. State the contradiction and conclude.

Example. Given that n is an integer, prove by contradiction that if n^2 is odd then n is odd.

Model answer.

Assume that n^2 is odd and n is even. (B1)

Then $n=2k$ for some integer k, so $n^2=4k^2$ (M1)

$=2(2k^2)$, which is even. (A1)

This contradicts n^2 being odd, so if n^2 is odd then n is odd. (A1)

Prove that a number is irrational 5 marks **Grade A**

1. Assume the number equals $\frac{a}{b}$ with a, b integers with no common factors.
2. Rearrange to an equation in integers, e.g. $a^3=2b^3$
3. Deduce a is even, substitute $a=2c$ and deduce b is even.
4. State the contradiction (common factor 2) and conclude.

Example. Prove by contradiction that $\sqrt[3]{2}$ is irrational.

Model answer.

Assume $\sqrt[3]{2}=\frac{a}{b}$, where a and b are integers with no common factors. (B1)

Then $a^3=2b^3$, so a^3 is even, so a is even (an odd number cubed is odd). (M1)

Let $a=2c$: $8c^3=2b^3$, so $b^3=4c^3$ (M1)

So b^3 is even, and so b is even. (A1)

So a and b share a factor of 2, which contradicts the assumption. Hence $\sqrt[3]{2}$ is irrational. (A1)

Prove there are no integer solutions 4 to 5 marks **Grade A***

1. Assume there are integers that satisfy the equation.
2. Factorise, e.g. $x^2 - y^2 = (x - y)(x + y)$
3. Use odd/even (or other) properties of the factors.
4. Show every case is impossible and conclude.

Example. Prove by contradiction that there are no integers x and y such that $x^2 - y^2 = 10$

Model answer.

Assume there are integers x and y with $x^2 - y^2 = 10$. (B1)

Then $(x - y)(x + y) = 10$. (M1)

$(x + y) + (x - y) = 2x$ is even, so $x + y$ and $x - y$ are both even or both odd. (M1)

Both odd: the product is odd, but 10 is even. Both even: the product is a multiple of 4, but 10 is not. (A1)

Every case is impossible, which is a contradiction, so no such integers exist. (A1)

Prove there are infinitely many primes (bookwork) 4 marks **Grade A**

1. Assume there are finitely many primes and list them.
2. Define N as their product plus 1.
3. Show no prime in the list divides N
4. Explain why this gives a prime not on the list, and conclude.

Example. Prove by contradiction that there are infinitely many prime numbers.

Model answer.

Assume there are finitely many primes, $p_1, p_2, \dots, p_n$. (B1)

Let $N = p_1 p_2 \cdots p_n + 1$. (M1)

Dividing N by any of $p_1, \dots, p_n$ leaves remainder 1, so none of them is a factor of N. (A1)

$N > 1$, so N is prime or has a prime factor not in the list, which contradicts the list containing every prime. So there are infinitely many primes. (A1)

Don't lose marks

- Not stating the assumption: the first mark is usually for it.
- Forgetting 'no common factors' in the irrationality proof, so the ending has nothing to contradict.
- Claiming $N = p_1 p_2 \cdots p_n + 1$ must be prime. It need not be; it only has a prime factor that is not in the list.

- Negating wrongly: the opposite of 'there is no largest even number' is 'there is a largest even number', not 'there is a smallest even number'.
- Stopping at the contradiction without saying what it means for the original statement.

More tips

Memory tricks

- Rhythm: Assume, Argue, Contradiction, Conclude.
- In irrationality proofs the contradiction is almost always 'so a and b have a common factor'.
- In integer-equation proofs, factorise and look at odd and even.
- Write 'with no common factors' in your first line: without it there is nothing to contradict later.

Exam technique

- Begin with 'Assume, for contradiction, that ...' and end with 'This contradicts ..., so ...'.
- Say exactly what is contradicted, e.g. 'this contradicts a and b having no common factors'.
- Learn the $\sqrt{2}$ and infinitely-many-primes proofs thoroughly: they can be asked as bookwork.
- If told 'you may assume $\sqrt{2}$ is irrational', use it: the contradiction will be that $\sqrt{2}$ would be rational.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Write the opening assumption correctly**
Assume the negation of the statement, e.g. for 'there is no greatest odd number' assume there is a greatest odd number, N

Grade B — **Prove simple number results by contradiction**
e.g. to prove 'if n^2 is odd then n is odd', assume n is even and show n^2 is even.

Grade A — **Prove that the square root of 2 is irrational**
Assume $\sqrt{2} = \frac{a}{b}$ in lowest terms, then show a and b are both even.

Grade A — **Prove that there are infinitely many primes**
Assume a finite list of all primes, multiply them together and add 1, and show this has a prime factor not in the list.

Grade A* — **Adapt the method to unfamiliar statements**
e.g. show there are no integers with $x^2 - y^2 = 10$, or that $\sqrt[3]{2}$ is irrational.

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