Proof by deduction, exhaustion and counter-example

P1.1a · Pure mathematics · revision notes

A proof is a chain of logical steps from facts you know to the statement you want. You need proof by deduction (general algebra that covers every case), proof by exhaustion (checking every possible case) and disproof by counter-example. Proof questions appear on Papers 1 and 2, usually for 3 to 5 marks, and always need a clear concluding statement.

Key facts

- Even: $2n$; odd: $2n+1$; consecutive: $n, n+1, n+2$ ($n \in \mathbb{Z}$)
- $(x-a)^2 \geq 0$ for all real x
- $\Rightarrow$ implies; $\Leftarrow$ is implied by; $\Leftrightarrow$ if and only if
- $n(n+1)$ is always even; $n(n+1)(n+2)$ is always a multiple of 6
- Counter-example: ONE case where the statement fails
- Multiple of k: show the expression $= k\times$ (integer)
- Rational number: $\frac{a}{b}$ with a, b integers, $b \neq 0$
- Every integer is $3k$, $3k+1$ or $3k+2$
- Exhaustion: the cases must cover every possibility

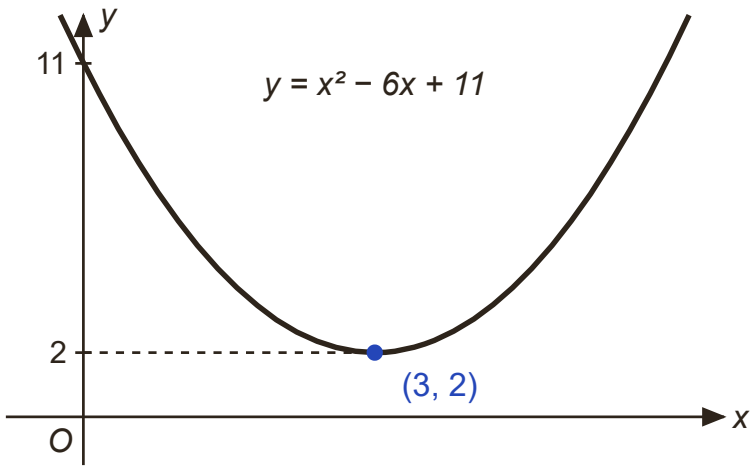
Notes

What a proof must do

- Start from facts that are known to be true: definitions, given information and standard results.
- Every step must follow from the one before. Only write $\Rightarrow$ ('implies') when the next line really follows.
- Checking a few examples is not a proof: examples only suggest a result, unless you check every possible case.
- Finish with a conclusion that repeats the statement, e.g. 'so $n^3 - n$ is a multiple of 6 for all integers n'.

Proof by deduction

- Write general numbers algebraically, with $n, m \in \mathbb{Z}$: even $2n$, odd $2n + 1$, consecutive integers n, $n + 1$, $n + 2$
- Use different letters for numbers that are not related: two odd numbers are $2n + 1$ and $2m + 1$
- To show something is a multiple of k, write it as $k\times$ (an expression that is an integer).
- To show an expression is always positive, complete the square:
  $x^2 - 6x + 11 = (x - 3)^2 + 2 \geq 2 > 0$, because a square is never negative.



The lowest point is (3, 2), so the curve never reaches the x-axis: x² − 6x + 11 > 0 for all x.

- For positive a and b, start from a square: $(\sqrt{a} - \sqrt{b})^2 \geq 0$ gives $a + b \geq 2\sqrt{ab}$ Grade A or above only

Proof by exhaustion

- Split all possibilities into a finite number of cases that together cover everything, then prove the statement in each case.
- Useful splits: n even or odd; $n = 3k$, $3k + 1$ or $3k + 2$; or a short finite list, e.g. every integer from 1 to 6.
- Say why your cases cover everything, and prove every case, not just some of them.

Disproof by counter-example

- One example where the statement is false is enough to disprove it.
- Show the calculation and then say clearly that it is a counter-example, so the statement is false.
- Good values to try: 0, 1, 2 (the only even prime), negative numbers and fractions.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Disprove a statement by counter-example

1 to 2 marks | Grade E

1. Try small values, including 1, 2, 0 and negative numbers, until the statement fails.
2. Show the calculation for that value.
3. State that this is a counter-example, so the statement is not true.

Example. Show that the statement '$2^n + 1$ is prime for every positive integer n' is false.

Model answer.

When $n = 3$, $2^3 + 1 = 9$ (M1)

$9 = 3 \times 3$ is not prime, so $n = 3$ is a counter-example and the statement is false. (A1)

Prove by deduction that a result holds for all integers

3 to 4 marks | Grade C

1. Write the numbers in general form, e.g. $2n + 1$ and $2n + 3$, and say that n is an integer.
2. Expand and simplify.
3. Rearrange into the form needed, e.g. $8\times$ (integer) $+2$
4. Write a conclusion that repeats the statement.

Example. Prove that the sum of the squares of any two consecutive odd numbers is 2 more than a multiple of 8.

Model answer.

Let the numbers be $2n + 1$ and $2n + 3$, where n is an integer. (B1)

$$(2n + 1)^2 + (2n + 3)^2$$

$$= 4n^2 + 4n + 1 + 4n^2 + 12n + 9$$

$$= 8n^2 + 16n + 10 \quad \text{(M1)}$$

$$= 8(n^2 + 2n + 1) + 2 \quad \text{(A1)}$$

$n^2 + 2n + 1$ is an integer, so the sum is 2 more than a multiple of 8. (A1)

Prove by exhaustion

3 to 4 marks | **Grade A**

1. Choose cases that cover every possibility, e.g. $n = 3k$, $3k+1$, $3k+2$, and say so.
2. Substitute each case and simplify.
3. Show the statement is true in every case.
4. Conclude for all integers.

Example. Prove by exhaustion that $n^2 + 1$ is not a multiple of 3 for any integer n

Model answer.

Every integer can be written as $3k$, $3k+1$ or $3k+2$, where k is an integer. (B1)

$n = 3k$: $n^2 + 1 = 9k^2 + 1$
$= 3(3k^2) + 1$ (M1)

$n = 3k+1$: $n^2 + 1 = 9k^2 + 6k + 2$
$= 3(3k^2 + 2k) + 2$

$n = 3k+2$: $n^2 + 1 = 9k^2 + 12k + 5$
$= 3(3k^2 + 4k + 1) + 2$ (A1 all three correct)

In every case the remainder on division by 3 is 1 or 2, so $n^2 + 1$ is never a multiple of 3. (A1)

Prove that an expression is always positive

3 marks | **Grade B**

1. Complete the square.
2. State that the squared bracket is never negative, for all real x
3. Conclude that the expression is at least a positive number, so it is always positive.

Example. Prove that $x^2 - 8x + 19 > 0$ for all real values of x

Model answer.

$x^2 - 8x + 19 = (x-4)^2 - 16 + 19$ (M1)

$= (x-4)^2 + 3$ (A1)

$(x-4)^2 \geq 0$ for all real x, so $(x-4)^2 + 3 \geq 3 > 0$. Hence $x^2 - 8x + 19 > 0$ for all real x. (A1)

Don't lose marks

- Using the same letter for two unrelated numbers, e.g. writing two odd numbers as $2n+1$ and $2n+1$, which only covers equal numbers.
- Checking a few values and calling it a proof. Unless every case is checked, examples prove nothing.

- Missing a case in exhaustion, e.g. forgetting zero or negative numbers when the statement says 'all integers'.
- Leaving $8n + 8$ instead of writing $8(n + 1)$ and saying that $n + 1$ is an integer.
- No conclusion: the final mark usually needs a statement linked to the question.

More tips

Memory tricks

- Stuck for a counter-example? Try 0, 1, 2, a negative number, a fraction and a large number.
- For 'multiple of k' proofs, your whole aim is to take k out as a factor of the entire expression.
- Product of two consecutive integers is always even: one of them must be even.
- Sense check: substitute a number such as $n = 2$ into your final factorised form and into the original expression.

Exam technique

- Define your variables at the start, e.g. 'Let n be an integer, so $2n + 1$ is odd'.
- End every proof with a conclusion using the words of the question, e.g. '... so it is a multiple of 8 for all integers n'.
- For 'disprove' questions, one clear counter-example with its working and a statement is enough.
- When the result is printed in the question, every step must be shown: examiners mark the argument, not the answer.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Disprove a statement with a counter-example**
Find one value that makes the statement false, show the working and say that it disproves the statement.

Grade D — **Write general even, odd and consecutive integers**
Use $2n$, $2n + 1$, and n, $n + 1$, where n is an integer, so the argument covers every case.

Grade C — **Prove a statement by exhaustion**
Split all possibilities into a finite set of cases (e.g. n even and n odd) and prove each one.

Grade C — **Prove an algebraic result by deduction**
Expand and factorise to show, for example, that an expression equals $8\times$ an integer.

Grade B — **Prove an expression is always positive**
Complete the square to get $(x - a)^2 + k$ with $k > 0$, then use $(x - a)^2 \geq 0$

Grade A — **Use remainders to cover every integer**
Write n as $3k$, $3k + 1$ or $3k + 2$ (or similar) and prove the result for each form.

Grade A* — **Construct a full proof in an unfamiliar context**
Choose a method, state every assumption, justify each step and finish with a precise conclusion.

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