Ladders, tilting and limiting equilibrium

M9.1b · Mechanics · revision notes

Limiting cases of equilibrium for rigid bodies: a beam on the point of tilting about a support, and rods and ladders resting against walls, pegs and rough ground, on the point of slipping. You combine resolving, taking moments and the friction model $F \leqslant \mu R$

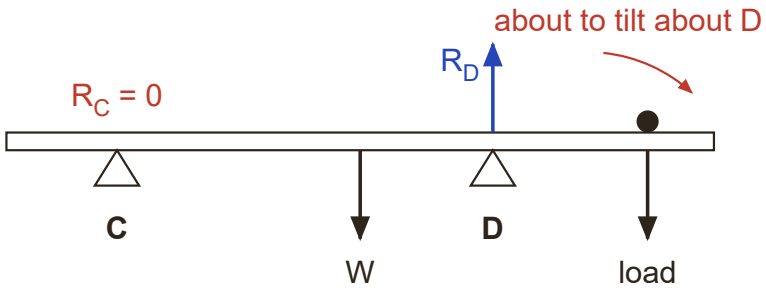
Key facts

- On the point of tilting about D: the reaction at the other support = 0
- Rough contact: $F \leqslant \mu R$; limiting: $F = \mu R$
- A horizontal force at distance d along the rod has moment arm $d \sin\theta$
- Smooth wall or peg: reaction perpendicular to the wall or rod, no friction
- Rod at θ to the horizontal: a vertical force at distance d has moment arm $d \cos\theta$
- Three equations: resolve ↑, resolve →, take moments

Notes

Tilting

- A beam resting on two supports is **on the point of tilting** about one support when the reaction at the other support becomes zero.

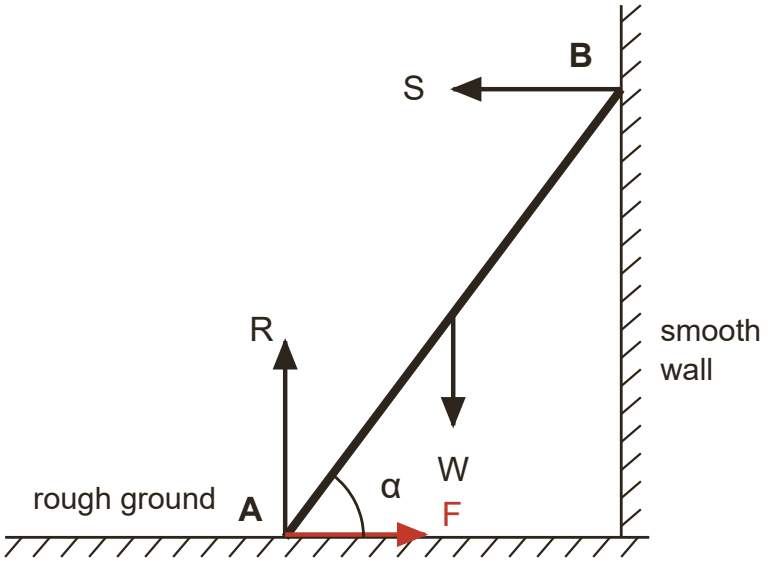


On the point of tilting about D: $R_C = 0$, so take moments about D

- Take moments about the support it tilts about, with the other reaction set to 0. This gives the greatest load, or the furthest point at which a person can stand.
- At the point of tilting the beam is still in equilibrium, so the usual equations still hold.

Rods and ladders at an angle

- Smooth wall: the reaction is perpendicular to the wall (horizontal), with no friction. Smooth peg: the reaction is perpendicular to the rod.
- Rough ground: a normal reaction R (vertical) and friction F (horizontal), acting towards the wall if the foot of the ladder tends to slide away from it.



Ladder forces: R and F at the ground, S at the smooth wall, W at the midpoint

- Rough wall: there is also friction along the wall, acting upwards if the top of the ladder tends to slide down.
- For a rod inclined at θ to the horizontal, a vertical force at distance d along the rod from the pivot has moment (force) × $d\cos\theta$; a horizontal force has moment (force) × $d\sin\theta$
- Taking moments about the foot of a ladder removes both R and F from the equation.

Limiting equilibrium

- On the point of slipping, $F = \mu R$ at each rough contact that is about to slip. Otherwise $F \leqslant \mu R$
- Method: resolve horizontally, resolve vertically, take moments (usually about the foot), then use $F = \mu R$ or $F \leqslant \mu R$
- 'Least possible value of μ': find F and R, then $F \leqslant \mu R$ gives $\mu \geqslant \dfrac{F}{R}$
- As a person climbs a ladder, the friction needed at the foot increases; the ladder slips when the friction needed would be more than μR
- Modelling: the ladder is a uniform rod (rigid, weight at the midpoint), the person is a particle, and the ladder lies in a vertical plane perpendicular to the wall.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Beam on the point of tilting

4 to 6 marks **Grade C**

1. Decide which support the beam would tilt about.
2. Set the reaction at the other support to zero.
3. Take moments about the tilting support and solve for the unknown distance or mass.

Example. A uniform plank AB, of length 6 m and mass 30 kg, rests horizontally on two supports at C and D, where $AC = 1$ m and $DB = 1.5$ m. A child of mass 40 kg walks along the plank from C towards B. The child is modelled as a particle. Find the distance from B of the child when the plank is on the point of tilting.

Model answer.

On the point of tilting about D, the reaction at C is zero (B1)

$AD = 4.5$ m and the plank's weight acts 3 m from A, so 1.5 m from D

Moments about D: $30g \times 1.5 = 40g \times y$ (M1 A1), so $y = 1.125$ m beyond D (A1)

Distance from B $= 1.5 - 1.125$

$= 0.375$ m (A1)

Ladder against a smooth wall: find μ

6 to 9 marks **Grade B**

1. Draw the forces: the weight at the midpoint, R and F at the foot, and S horizontal at the wall.
2. Resolve vertically (R = weight) and horizontally ($F = S$).
3. Take moments about the foot to find S
4. Limiting equilibrium: $F = \mu R$

Example. A uniform ladder AB, of length 5 m and mass 20 kg, rests with its foot A on rough horizontal ground and its top B against a smooth vertical wall. The ladder makes an angle α with the ground, where $\tan\alpha = \frac{4}{3}$, and lies in a vertical plane perpendicular to the wall. The ladder is in limiting equilibrium. Find the coefficient of friction between the ladder and the ground.

Model answer.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$

Vertically: $R = 20g$ (B1); horizontally: $F = S$ (B1)

Moments about A: $S \times 5\sin\alpha = 20g \times 2.5\cos\alpha$ (M1 A1), so $4S = 30g$ and $S = 7.5g$ (A1)

$F = \mu R$: $7.5g = \mu \times 20g$ (M1), so $\mu = \frac{3}{8}$ (A1)

Person climbing a ladder: when does it slip? 7 to 9 marks **Grade A**

1. Resolve vertically: R = total weight of the ladder and the person.
2. About to slip: $F = \mu R$, and horizontally $S = F$
3. Take moments about the foot, with the person at distance x up the ladder.
4. Solve for x and check that it is less than the length of the ladder.

Example. A uniform ladder, of length 4 m and mass 10 kg, rests at 60° to the horizontal with its foot on rough horizontal ground and its top against a smooth vertical wall. The ladder lies in a vertical plane perpendicular to the wall. The coefficient of friction between the ladder and the ground is 0.4. A man of mass 70 kg climbs the ladder. Modelling the man as a particle, find how far up the ladder he can climb before the ladder slips.

Model answer.

Vertically: $R = 80g$ (B1)

About to slip: $F = 0.4 \times 80g = 32g$ (M1), and horizontally $S = F = 32g$ (A1)

Moments about the foot:
$$S \times 4\sin 60^\circ = 10g \times 2\cos 60^\circ + 70g \times x\cos 60^\circ$$
(M1 A1 A1)

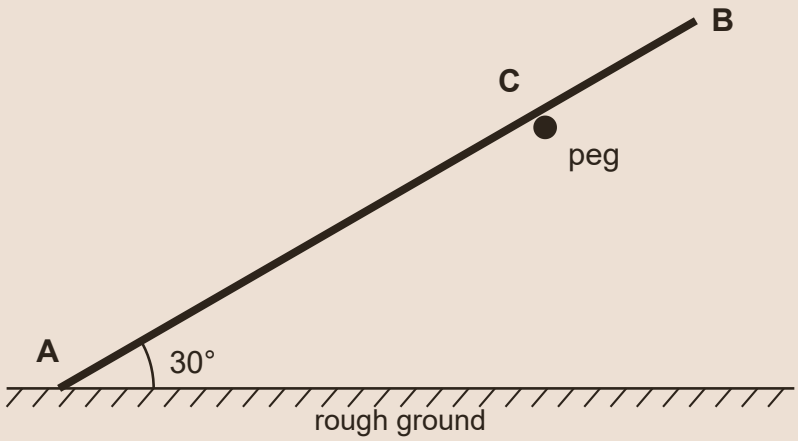
Dividing by g: $128\sin 60^\circ = 10 + 35x$, so $x = 2.88$ m (3 s.f.) (A1)

Rod resting on a smooth peg

7 to 9 marks | **Grade A***

1. The peg is smooth, so its force N is perpendicular to the rod.
2. Take moments about the end on the ground to find N
3. Resolve horizontally and vertically, using the components $N\sin\theta$ and $N\cos\theta$
4. Limiting equilibrium: $F = \mu R$

Example. A uniform rod AB, of length 2 m and weight W newtons, rests with its end A on rough horizontal ground and with the point C of the rod, where $AC = 1.5$ m, resting against a small smooth peg. The rod is inclined at 30° to the horizontal, in a vertical plane perpendicular to the peg, and is in limiting equilibrium. Find the coefficient of friction between the rod and the ground.



Model answer.

Moments about A: $N \times 1.5 = W \times 1 \times \cos 30°$ (M1 A1), so $N = \dfrac{W}{\sqrt{3}}$ (A1)

Horizontally: $F = N\sin 30° = \dfrac{W}{2\sqrt{3}}$ (M1)

Vertically: $R + N\cos 30° = W$, so $R = W - \dfrac{W}{2} = \dfrac{W}{2}$ (M1 A1)

$F = \mu R$: μ

$$= \frac{W}{2\sqrt{3}} \div \frac{W}{2}$$

$$= \frac{1}{\sqrt{3}}$$

$$= 0.577 \quad \text{(3 s.f.)}$$

(M1 A1)

Don't lose marks

- Putting friction at a smooth wall or a smooth peg.
- Drawing the peg's reaction vertically instead of perpendicular to the rod.

- Using $d\sin\theta$ instead of $d\cos\theta$ for the moment arm of a vertical force, when θ is the angle with the horizontal.
- Using $F = \mu R$ when the rod is not on the point of slipping.
- Setting the wrong reaction to zero in a tilting question.
- Leaving out the person's weight, or using the wrong distance for it.

More tips

Memory tricks

- Uniform ladder, smooth wall, rough ground, limiting equilibrium: $\mu = \dfrac{1}{2\tan\theta}$, where θ is the angle with the ground. Use it to check, but derive it in the exam.
- Moments about the foot of a ladder remove two unknowns at once (R and F).
- If $\tan\theta = \dfrac{4}{3}$, then $\sin\theta = \dfrac{4}{5}$ and $\cos\theta = \dfrac{3}{5}$
- When the weight is given as W, keep it as a letter: it usually cancels.

Exam technique

- Draw a large diagram showing every force at every contact before writing any equations.
- Write the key statement: 'on the point of tilting about D, so $R_C = 0$' or 'limiting equilibrium, so $F = \mu R$'. These often earn marks on their own.
- Explain modelling words when asked: 'the ladder is modelled as a rod' means it is rigid and its thickness is ignored.
- For 'least value of μ', show the inequality $F \leqslant \mu R$ before giving the value.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Take moments for forces on an inclined rod**
For a rod at θ to the horizontal, a vertical force at distance d along it has moment arm $d\cos\theta$; a horizontal force has $d\sin\theta$

Grade C — **Recognise when a beam is about to tilt**
On the point of tilting about one support, the reaction at the other support is zero.

Grade C — **Find limits on loads without tilting**
Take moments about the support it would tilt about, with the other reaction set to zero.

Grade B — **Draw the forces on a ladder correctly**
Smooth wall: horizontal reaction only; rough ground: normal reaction up and friction towards the wall.

Grade B — **Find μ for a ladder in limiting equilibrium**
Resolve both ways, take moments about the foot, then use $F = \mu R$

Grade A — **Solve ladder problems with a person climbing**
Put the person at distance x up the ladder and find the value of x at which it is about to slip.

Grade A* — **Solve rod-on-peg and rough-wall problems**
A smooth peg's reaction is perpendicular to the rod; at every rough contact that is about to slip, $F = \mu R$

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