Projectiles

M7.5 · Mechanics · revision notes

A projectile is a particle moving freely under gravity in a vertical plane. You treat the horizontal motion (constant velocity) and the vertical motion (constant acceleration g downwards) separately, linked by the time. Questions ask for the time of flight, range, greatest height, speed and direction, and may ask you to derive the standard results or the equation of the path.

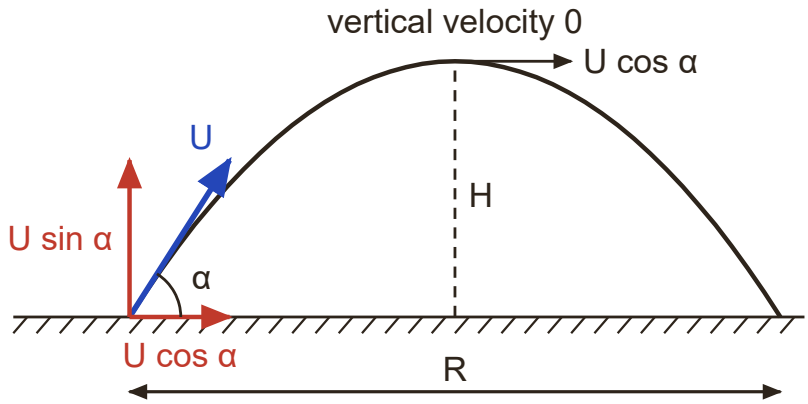
Key facts

- Horizontal: $x = (U\cos\alpha)t$, constant velocity
- Greatest height: $v_y = 0$
- Greatest range on level ground when $\alpha = 45^\circ$
- Speed $= \sqrt{v_x^2 + v_y^2}$; direction $\tan\theta = \dfrac{v_y}{v_x}$
- Vertical (up positive): $v_y = U\sin\alpha - gt$, $y = (U\sin\alpha)t - \dfrac{1}{2}gt^2$
- Level ground: $T = \dfrac{2U\sin\alpha}{g}$, $R = \dfrac{U^2\sin 2\alpha}{g}$, $H = \dfrac{U^2\sin^2\alpha}{2g}$
- Path: $y = x\tan\alpha - \dfrac{gx^2}{2U^2}(1+\tan^2\alpha)$

Notes

The model

- The projectile is a particle moving freely under gravity: air resistance is ignored, g is constant and the motion is in a vertical plane.
- The horizontal and vertical motions are independent. The time t is the same for both and links them.
- Horizontally there is no force, so the velocity is constant: $x = (U\cos\alpha)t$
- Vertically the acceleration is g downwards. Taking up as positive: $v_y = U\sin\alpha - gt$ and $y = (U\sin\alpha)t - \frac{1}{2}gt^2$



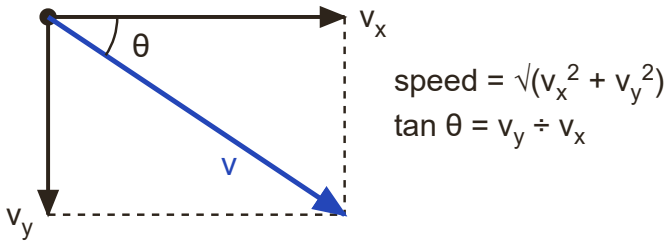
Split U into U cos α (constant) and U sin α (acted on by g)

Key results on level ground

- At the greatest height the vertical component of velocity is zero. The velocity there is horizontal, $U\cos\alpha$, and this is the least speed.
- Time of flight: set $y = 0$, giving $T = \dfrac{2U\sin\alpha}{g}$
- Range: $R = U\cos\alpha \times T = \dfrac{U^2\sin 2\alpha}{g}$, which is greatest when $\alpha = 45°$
- Greatest height: $H = \dfrac{U^2\sin^2\alpha}{2g}$, reached after half the time of flight.
- These results only hold when the projectile lands at the same height as it started. From a cliff or onto a roof, use suvat vertically with the correct (possibly negative) displacement.
- Unless you are told you may quote them, derive these results from suvat.

Speed, direction, vectors and the path

- Speed at time t: $\sqrt{v_x^2 + v_y^2}$. Direction: $\tan\theta = \dfrac{v_y}{v_x}$, above the horizontal if $v_y > 0$ and below it if $v_y < 0$



Combine the components to get the speed and direction

- Vector form, with $\mathbf{i}$ horizontal and $\mathbf{j}$ vertically up: $\mathbf{a} = -g\mathbf{j}$, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$, $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \dfrac{1}{2}\mathbf{a}t^2$
- Equation of the path: substitute $t = \dfrac{x}{U\cos\alpha}$ into the vertical equation to get $y = x\tan\alpha - \dfrac{gx^2}{2U^2\cos^2\alpha}$, a parabola. Writing $\dfrac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$ gives a quadratic in $\tan\alpha$

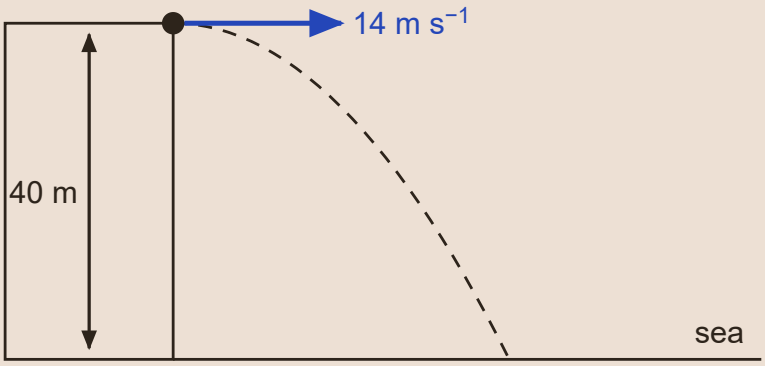
How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Horizontal projection from a height 4 to 7 marks **Grade C**

1. Vertically: $u = 0$, $a = 9.8$ (taking down as positive) and s = the height. Find t
2. Horizontally: distance = horizontal speed × t
3. For the speed on landing, find the vertical velocity gt and combine it with the horizontal speed.

Example. A ball is thrown horizontally with speed 14 m s^{-1} from the top of a vertical cliff, 40 m above the sea. The ball is modelled as a particle moving freely under gravity.



(a) Find the time taken for the ball to reach the sea.
(b) Find the horizontal distance travelled by the ball.
(c) Find the speed of the ball as it reaches the sea.

Model answer.

(a) Vertically, down positive: $40 = \frac{1}{2}(9.8)t^2$ (M1), $t = \frac{20}{7} = 2.86$ s (3 s.f.) (A1)

(b) $14 \times \frac{20}{7}$ (M1) = 40 m (A1)

(c) Vertical speed $= 9.8 \times \frac{20}{7} = 28$ m s^{-1} (M1); speed $= \sqrt{14^2 + 28^2}$ (M1) = 31 m s^{-1} (2 s.f.) or 31.3 m s^{-1} (3 s.f.) (A1)

Projection at an angle: greatest height, time of flight, range 5 to 8 marks **Grade C**

1. Resolve: horizontal $U\cos\alpha$, vertical $U\sin\alpha$, using exact values if $\tan\alpha$ is given.
2. Greatest height: vertical $v = 0$, use $v^2 = u^2 + 2as$
3. Time of flight: vertical $s = 0$, use $s = ut + \frac{1}{2}at^2$
4. Range: horizontal speed × time of flight.

Example. A particle is projected from a point O on horizontal ground with speed 21 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. The particle moves freely under gravity.

(a) Find the greatest height of the particle above the ground.

(b) Find the time for which the particle is in the air.

(c) Find the distance from O to the point where the particle lands.

Model answer.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$, so the vertical component is 16.8 m s^{-1} and the horizontal component is 12.6 m s^{-1} (B1)

(a) $0 = 16.8^2 - 2(9.8)h$ (M1), $h = 14.4$ m (A1)

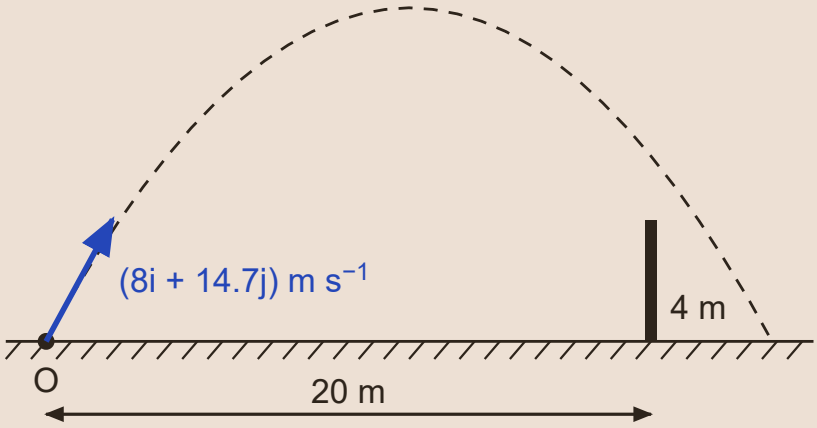
(b) $0 = 16.8t - 4.9t^2$ (M1), $t = \frac{24}{7} = 3.43$ s (3 s.f.) (A1)

(c) $12.6 \times \frac{24}{7}$ (M1) = 43.2 m (A1)

Time above a height, or clearing an obstacle (vector form) 5 to 8 marks **Grade B**

1. Write the horizontal and vertical displacements in terms of t, using the components of the velocity vector.
2. For 'at least h m above the ground', solve vertical displacement $= h$; the time between the two roots is the answer.
3. For an obstacle at horizontal distance d, find t from the horizontal motion and substitute it into the vertical equation.

Example. [$\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upward unit vectors.] A ball is kicked from a point O on horizontal ground with velocity $(8\mathbf{i} + 14.7\mathbf{j})$ m s^{-1}. The ball is modelled as a particle moving freely under gravity.



(a) Find the length of time for which the ball is at least 9.8 m above the ground.
(b) A vertical fence of height 4 m stands 20 m horizontally from O, in the vertical plane of the ball's motion. Find the height by which the ball clears the fence.

Model answer.

(a) $14.7t - 4.9t^2 = 9.8$ (M1), so $t^2 - 3t + 2 = 0$, giving $t = 1$ or $t = 2$ (A1); time = 1 s (A1)

(b) $8t = 20$, so $t = 2.5$ (M1); height $= 14.7(2.5) - 4.9(2.5)^2$ (M1) = 6.125 m (A1); the ball clears the fence by 2.1 m (2 s.f.) or 2.13 m (3 s.f.) (A1)

Derive the equation of the path; find angles of projection 6 to 9 marks **Grade A***

1. Write $x = (U\cos\alpha)t$ and $y = (U\sin\alpha)t - \frac{1}{2}gt^2$
2. Make t the subject of the horizontal equation and substitute into the vertical one.
3. Use $\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$
4. Substitute the point's coordinates to get a quadratic in $\tan\alpha$; both roots may be valid.

Example. A particle is projected from a point O with speed U m s^{-1} at an angle α above the horizontal and moves freely under gravity. At time t seconds it is x m horizontally and y m vertically from O.

(a) Show that $y = x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$.

(b) Given that $U = 14$ and that the particle passes through the point which is 10 m horizontally from O and 5 m above O, find the two possible values of α

Model answer.

(a) $x = (U\cos\alpha)t$, so $t = \frac{x}{U\cos\alpha}$ (M1); $y = (U\sin\alpha)t - \frac{1}{2}gt^2$ (M1)

$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$ (M1) $= x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$ (A1*)

(b) $5 = 10\tan\alpha - \frac{9.8\times 100}{2\times 196}(1 + \tan^2\alpha)$ (M1), i.e.

$5 = 10\tan\alpha - 2.5(1 + \tan^2\alpha)$, so $\tan^2\alpha - 4\tan\alpha + 3 = 0$ (A1)

$\tan\alpha = 1$ or $\tan\alpha = 3$ (M1), so $\alpha = 45°$ or $\alpha = 71.6°$ (A1)

Don't lose marks

- Putting g into the horizontal motion, or using U instead of $U\cos\alpha$ horizontally.
- Saying the velocity is zero at the top: only the vertical component is zero.
- Using the level-ground results when the projectile starts or lands at a different height.
- Sign errors: with up as positive, a point 40 m below the start has $y = -40$
- Giving only the vertical component when asked for the speed.
- Rounding t early and then using the rounded value for the horizontal distance.

More tips

Memory tricks

- Time is the link: find t from one direction and use it in the other.
- On level ground the motion is symmetrical: time up = time down, and the landing speed equals the projection speed.
- Horizontal projection from height h: $t = \sqrt{\dfrac{2h}{g}}$
- If $\tan\alpha = \dfrac{3}{4}$, then $\sin\alpha = \dfrac{3}{5}$ and $\cos\alpha = \dfrac{4}{5}$; if $\tan\alpha = \dfrac{5}{12}$, then $\sin\alpha = \dfrac{5}{13}$ and $\cos\alpha = \dfrac{12}{13}$

Exam technique

- Head your working 'Horizontally' and 'Vertically' and state your positive direction.
- If asked to 'show that' a result, start from suvat in each direction; do not quote the result.
- Keep exact values (such as $\dfrac{20}{7}$) until the end, and give answers to 2 or 3 s.f. when $g = 9.8$ is used.
- Typical assumptions to state: the ball is a particle; air resistance is negligible. A typical refinement: include air resistance.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Resolve the initial velocity into components**
The horizontal component is $U\cos\alpha$ and the vertical component is $U\sin\alpha$

Grade C — **Solve horizontal projection problems**
Vertically $u = 0$ and the acceleration is g; horizontally the velocity is constant; the time links them.

Grade C — **Find time of flight, range and greatest height**
At the greatest height the vertical velocity is zero; on landing the vertical displacement is known.

Grade B — **Find the speed and direction at a given time**
Combine the horizontal and vertical velocity components using Pythagoras and $\tan\theta$

Grade B — **Use the vector form of projectile motion**
With $\mathbf{j}$ vertically up, $\mathbf{a} = -g\mathbf{j}$, so $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and $\mathbf{r} = \mathbf{u}t + \dfrac{1}{2}\mathbf{a}t^2$

Grade A — **Derive general projectile formulae**
Find the time of flight, range and greatest height in terms of U, α and g

Grade A* — **Find angles of projection through a point**
Use the equation of the path with $\sec^2\alpha = 1 + \tan^2\alpha$ to get a quadratic in $\tan\alpha$

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