Variable acceleration using calculus

M7.4 · Mechanics · revision notes

When the acceleration varies with time you cannot use suvat: you differentiate and integrate instead. Questions give displacement, velocity or acceleration as a function of t, in one dimension or as an $\mathbf{i}$, $\mathbf{j}$ vector, and ask for velocities, accelerations, times at rest, greatest values and distances travelled.

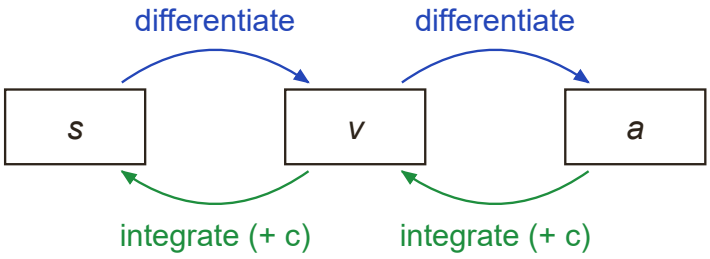
Key facts

- $v = \dfrac{ds}{dt}$, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$
- Vectors: $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$
- Displacement from t_1 to t_2: $\int_{t_1}^{t_2} v\,dt$
- $v = \int a\,dt$, $s = \int v\,dt$ (+ constant)
- At rest: $v = 0$. Greatest or least velocity: $a = 0$
- Distance: split where $v = 0$ and add the magnitudes

Notes

The calculus links

- Displacement s → differentiate → velocity v → differentiate → acceleration a
- Acceleration → integrate → velocity → integrate → displacement.

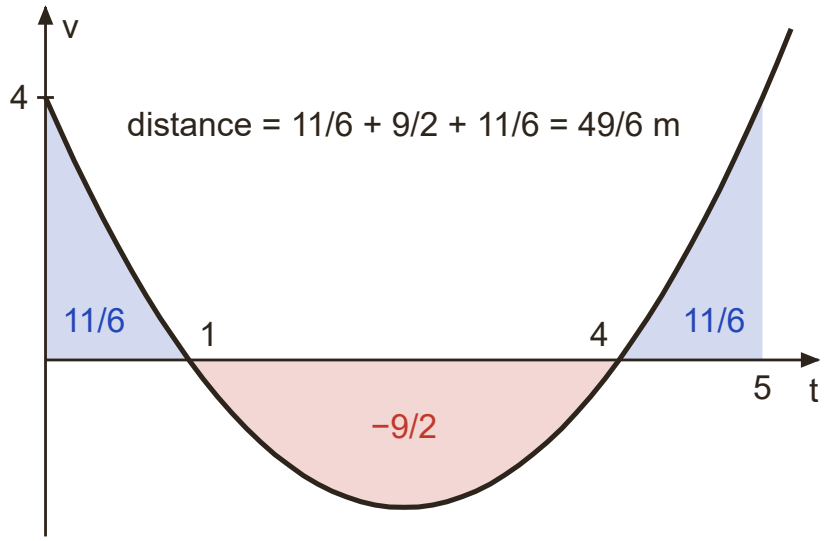


Differentiate going right; integrate going left

- $v = \dfrac{ds}{dt}$, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$, $v = \int a\,dt$, $s = \int v\,dt$
- Dot notation: $\dot{s} = v$ and $\ddot{s} = a$ (each dot means differentiate once with respect to time).
- Use calculus whenever the acceleration is not constant, e.g. $a = 6 - 2t$. Suvat is only for constant acceleration.

Constants and key moments

- Every indefinite integral needs a constant. Find it from the initial conditions: 'starts from rest at O' gives $v = 0$ and $s = 0$ when $t = 0$
- Instantaneously at rest: $v = 0$
- Greatest or least velocity: $\dfrac{dv}{dt} = 0$, i.e. $a = 0$. Greatest or least displacement: $v = 0$
- The displacement between $t = t_1$ and $t = t_2$ is $\int_{t_1}^{t_2} v\,dt$
- If v changes sign during the interval, the particle changes direction. Find where $v = 0$, work out the displacement for each part, and add their magnitudes to get the total distance.



v = (t − 1)(t − 4) changes sign at t = 1 and t = 4: add the sizes of the separate areas

- The functions can come from anywhere in the Pure course: polynomials, e^{kt}, $\sin kt$, $\cos kt$

Vectors in two dimensions

- If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are functions of t, then $\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$
- Integrate each component separately. The constant is a vector $p\mathbf{i} + q\mathbf{j}$, found from the initial position or velocity.
- Speed $= |\mathbf{v}|$. Moving parallel to $\mathbf{i}$: the $\mathbf{j}$ component of $\mathbf{v}$ is zero. Moving parallel to $\mathbf{j}$: the $\mathbf{i}$ component of $\mathbf{v}$ is zero.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Differentiate to find velocity and acceleration

4 to 6 marks | **Grade D**

1. Differentiate s to get v, and v to get a
2. 'At rest' means solve $v = 0$; reject negative times.
3. Substitute the given time and state the units.

Example. A particle P moves along a straight line. At time t seconds, $t \geqslant 0$, its displacement from a fixed point O on the line is s metres, where $s = t^3 - 6t^2 + 9t$.
(a) Find the values of t when P is instantaneously at rest.
(b) Find the acceleration of P when $t = 4$

Model answer.

(a) $v = 3t^2 - 12t + 9$ (M1 A1); $3(t-1)(t-3) = 0$ (M1), so $t = 1$ and $t = 3$ (A1)

(b) $a = 6t - 12$ (M1); when $t = 4$, $a = 12$ m s^{-2} (A1)

Integrate from acceleration, using initial conditions

5 to 8 marks | **Grade C**

1. Integrate a to get v, including $+c$
2. Use the velocity at a known time to find c
3. For the greatest velocity, solve $a = 0$ and substitute into v
4. Integrate v for the displacement, finding the new constant in the same way.

Example. A particle moves along a straight line. At time t seconds its acceleration is $(6 - 2t)$ m s^{-2}. When $t = 0$ the particle is at the point O and its velocity is 4 m s^{-1}.
(a) Find the velocity of the particle at time t.
(b) Find the greatest velocity of the particle.
(c) Find the displacement of the particle from O when $t = 3$

Model answer.

(a) $v = 6t - t^2 + c$ (M1); $v = 4$ when $t = 0$ gives $c = 4$, so $v = 4 + 6t - t^2$ (A1)

(b) $a = 0$ when $t = 3$, a maximum since $\dfrac{da}{dt} = -2 < 0$ (M1); $v = 4 + 18 - 9 = 13$ m s^{-1} (A1)

(c) $s = 4t + 3t^2 - \dfrac{t^3}{3}$, with no constant as $s = 0$ when $t = 0$ (M1 A1); when $t = 3$, $s = 12 + 27 - 9 = 30$ m (A1)

Total distance when the particle changes direction

5 to 6 marks | **Grade A**

1. Solve $v = 0$ to find the times in the interval when the particle turns round.
2. Integrate v to get s (or use a definite integral for each part).
3. Find the displacement for each part and add the magnitudes.

Example. A particle moves along a straight line so that at time t seconds its velocity is v m s^{-1}, where $v = t^2 - 5t + 4$, $t \geqslant 0$. Find the total distance travelled by the particle in the first 5 seconds.

Model answer.

$v = (t-1)(t-4)$
$= 0$ when $t = 1$ and $t = 4$ (M1)

$s = \dfrac{t^3}{3} - \dfrac{5t^2}{2} + 4t$, taking $s = 0$ when $t = 0$ (M1)

$s(1) = \dfrac{11}{6}$, $s(4) = -\dfrac{8}{3}$, $s(5) = -\dfrac{5}{6}$ (A1)

Distance $= \dfrac{11}{6} + \left(\dfrac{11}{6} + \dfrac{8}{3}\right) + \left(\dfrac{8}{3} - \dfrac{5}{6}\right)$ (M1) $= \dfrac{49}{6} = 8.17$ m (3 s.f.) (A1)

Vector calculus in two dimensions

6 to 10 marks | **Grade A**

1. Differentiate or integrate the $\mathbf{i}$ and $\mathbf{j}$ components separately.
2. Include a constant vector when integrating and find it from the given position or velocity.
3. Use the condition given: 'parallel to $\mathbf{j}$' means the $\mathbf{i}$ component of $\mathbf{v}$ is zero.
4. Find magnitudes using Pythagoras.

Example. A particle P moves in a horizontal plane. At time t seconds, $t \geqslant 0$, its velocity is $\mathbf{v}$ m s^{-1}, where $\mathbf{v} = (3t^2 - 12)\mathbf{i} + (4t - 2)\mathbf{j}$. When $t = 0$, P is at the point with position vector $(2\mathbf{i} + \mathbf{j})$ m.

(a) Find the magnitude of the acceleration of P when $t = 2$.

(b) Find the position vector of P at time t.

(c) Find the position vector of P at the instant when it is moving parallel to $\mathbf{j}$

Model answer.

(a) $\mathbf{a} = 6t\mathbf{i} + 4\mathbf{j}$ (M1); when $t = 2$, $\mathbf{a} = 12\mathbf{i} + 4\mathbf{j}$ and $|\mathbf{a}| = \sqrt{160} = 12.6$ m s^{-2} (3 s.f.) (M1 A1)

(b) $\mathbf{r} = (t^3 - 12t)\mathbf{i} + (2t^2 - 2t)\mathbf{j} + \mathbf{c}$ (M1 A1), with $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$, so

$\mathbf{r} = (t^3 - 12t + 2)\mathbf{i} + (2t^2 - 2t + 1)\mathbf{j}$ (A1)

(c) $3t^2 - 12 = 0$, so $t = 2$ (M1 A1); $\mathbf{r} = (-14\mathbf{i} + 5\mathbf{j})$ m (A1)

Don’t lose marks

- Using suvat when the acceleration depends on t
- Leaving out the constant of integration, or assuming it is zero without checking.
- Using $\int v\,dt$ across a change of direction as the distance travelled.
- Differentiating when you should integrate, or the other way round.
- Forgetting the initial position vector when integrating a velocity vector.

More tips

Memory tricks

- Differentiate going down the chain $s \to v \to a$; integrate going back up $a \to v \to s$
- Check a constant of integration by substituting $t = 0$ back in: you should get the initial value you were given.
- Use your calculator's definite integral function to check an answer, but show the integration in your working.
- A quick sketch of v against t shows where the particle changes direction.

Exam technique

- Show the differentiated or integrated expression before substituting numbers: the method marks depend on it.
- Check the domain, usually $t \geqslant 0$, and reject negative times.
- State units: m for displacement, m s^{-1} for velocity, m s^{-2} for acceleration.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Differentiate displacement to find velocity and acceleration**

$v = \dfrac{ds}{dt}$ and $a = \dfrac{dv}{dt}$

Grade C — **Integrate to find velocity or displacement**

Integrate, then use the initial conditions to find the constant of integration.

Grade C — **Find when a particle is at rest**

Solve $v = 0$; for the greatest or least velocity, solve $a = 0$

Grade B — **Differentiate and integrate vectors in terms of t**

Work on the $\mathbf{i}$ and $\mathbf{j}$ components separately; the constant of integration is a vector.

Grade A — **Use calculus with exponential or trigonometric functions**

Apply the Pure rules, e.g. $v = 4e^{-0.5t}$ gives $a = -2e^{-0.5t}$

Grade A — **Find the distance when the direction changes**

Split the time interval where $v = 0$, find each displacement and add their magnitudes.

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